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Refutation of the Commodity Exploitation Theorem

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A REFUTATION OF THE COMMODITY EXPLOITATION THEOREM

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ABSTRACT

This note is to show that the generalized commodity exploitation theorem put forward by Bowles and Gintis (1981) and Roemer (1982, 1986) is nothing but an alternative form of the Hawkins-Simon condition for a given technical data, and that it has nothing to do with exploitation. That is, the Hawkins-Simon condition means a mere possibility of an economic system to produce a surplus in each commodity, and as such does not guarantee the existence of positive profits. To consider exploitation or the existence of positive profits, we need to introduce prices at which unequal exchanges may be carried out. One more point is that labour is reproduced in a process which is supposed to earn no profit.

1. INTRODUCTION

Some economists, especially in the USA and Japan, argue that positive profits can accrue from “exploitation” of any one commodity chosen, and so it is useless to stick to the labour theory of value to explain profits based upon exploitation of labour as advocated by Marx. (See Bowles and Gintis (1981) and Roemer (1982, 1986).) This proposition is called the “commodity exploitation theorem”(CET), and not a few people think that this theorem puts an end to the fundamental Marxian theorem (FMT) established by Okishio (1963) and Morishima (1973). In this note, however, we show, using a simple Leontief model, that the CET

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is merely an alternative representation of the Hawkins-Simon condition, and simply says that a given linear economic model is productive enough to produce a surplus in each commodity. As such, the CET describes a *physical* condition, and has nothing to do with exploitation, a phenomenon of *social* relationship. Exploitation turns out after products are exchanged in the markets at stable or equilibrium prices as the economy dictates the participants.

We then prove that labour is exchanged for the real wage basket of commodities which is of less value whichever commodity in the basket is chosen as the numeraire or the standard of value. Thus, labour as a whole is exploited by the owners of products, the ‘capitalist class’ or the haves.

In Section 2, we restate the generalized commodity exploitation theorem, using the proof method in Fujita (2006), and Section 3 presents our propositions. Two numerical examples are given in the following Section 4. The final section contains some remarks, among which we describe how to generalize our results in a joint production model.

2. THE COMMODITY EXPLOITATION THEOREM RESTATED

Our model is a simple Leontief model of circulating capital. There exist n ($n \geq 1$) kinds of normal commodities and one type of labour. Let A represent a given $n \times n$ material input coefficient matrix, ℓ a given n -row labour input coefficient vector, and c a given n -column vector of consumption goods which is exchanged for a unit of labour force as the real wage basket. All the given matrix and the vectors are nonnegative.

We define an extended $(n + 1) \times (n + 1)$ matrix \mathbb{A} as

$$\mathbb{A} \equiv \begin{pmatrix} A & c \\ \ell & 0 \end{pmatrix}.$$

Let us make the following assumptions.

Assumption (HS). The matrix $(I - \mathbb{A})$ satisfies the Hawkins-Simon condition. (Hawkins and Simon (1949). Here, I is the $(n + 1) \times (n + 1)$ identity matrix.) This assumption means that the technology of our economy is so productive that it can produce a surplus in each commodity including labour force. Let $\mathbb{A}_{(i, i)}$ denote the $n \times n$ matrix obtained by deleting the i -th row and the i -th column from \mathbb{A} .

Now we consider the system of values in terms of i -th commodity. These values are defined by

$$(\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_{i-1}^{(i)}, \lambda_i^{(i)}, \lambda_{i+1}^{(i)}, \dots, \lambda_{n+1}^{(i)}) = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_{i-1}^{(i)}, 1, \lambda_{i+1}^{(i)}, \dots, \lambda_{n+1}^{(i)}) \cdot \mathbb{A}.$$

Note that on the LHS, the i -th entry in the variable vector is $\lambda_i^{(i)}$, while on the RHS the corresponding entry is unity. Here, the symbol $\lambda_j^{(i)}$ stands for the value of one unit of commodity j in terms of commodity i . For short, we call

this $\lambda_j^{(i)}$, ‘commodity i value of commodity j ’ like ‘labour value of bread’.
Let us define the following vectors:

$$\begin{aligned}\Lambda^{(i)} &\equiv (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_{i-1}^{(i)}, \lambda_i^{(i)}, \lambda_{i+1}^{(i)}, \dots, \lambda_{n+1}^{(i)}), \\ \Lambda_{[i]}^{(i)} &\equiv (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_{i-1}^{(i)}, 1, \lambda_{i+1}^{(i)}, \dots, \lambda_{n+1}^{(i)}), \\ \Lambda_{(i)}^{(i)} &\equiv (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_{i-1}^{(i)}, \lambda_{i+1}^{(i)}, \dots, \lambda_{n+1}^{(i)}), \text{ and} \\ \Lambda_{[i](n+1)}^{(i)} &\equiv (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_{i-1}^{(i)}, 1, \lambda_{i+1}^{(i)}, \dots, \lambda_n^{(i)}).\end{aligned}$$

(A pair of parentheses () in subscript implies the removal of the index inside, and the dimension gets one less.)

Then the above equation becomes

$$\Lambda^{(i)} = \Lambda_{[i]}^{(i)} \cdot \mathbb{A}. \quad (1)$$

This formulation is coherent with the usual definition of labour values when $i = n + 1$, i.e., labour is selected as the standard commodity of value, and $\lambda_j^{(i)}$ means the quantity of commodity i which is directly and indirectly required to produce one net unit of commodity j .

With a slight manipulation, eq.(1) is rewritten as

$$\Lambda_{[i]}^{(i)} = \Lambda_{[i]}^{(i)} \cdot \mathbb{A} + (0, 0, \dots, 0, 1 - \lambda_i^{(i)}, 0, \dots, 0), \text{ or} \quad (2)$$

where a possibly nonzero element in the second term on the RHS is in the i -th entry. Thus,

$$\Lambda_{[i]}^{(i)} = (0, 0, \dots, 0, 1 - \lambda_i^{(i)}, 0, \dots, 0) \cdot (I - \mathbb{A})^{-1}.$$

Denote by b_{ij} the (i, j) element of $(I - \mathbb{A})^{-1}$, then we have

$$\Lambda_{[i]}^{(i)} = (1 - \lambda_i^{(i)}) \cdot (b_{i1}, b_{i2}, \dots, b_{i, n+1}).$$

In particular, it follows

$$\begin{aligned}1 &= (1 - \lambda_i^{(i)}) \cdot b_{ii}, \text{ and} \\ \lambda_j^{(i)} &= (1 - \lambda_i^{(i)}) \cdot b_{ij}.\end{aligned}$$

These lead to

$$\lambda_i^{(i)} = \frac{b_{ii} - 1}{b_{ii}}, \text{ and} \quad (3)$$

$$\lambda_j^{(i)} = \frac{b_{ij}}{b_{ii}}. \quad (4)$$

By the Hawkins-Simon condition postulated in Assumption (HS),

$$(I - \mathbb{A})^{-1} = I + \mathbb{A} + \mathbb{A}^2 + \dots,$$

with the RHS converging. This implies

$$b_{kk} \geq 1 \text{ for any } 1 \leq k \leq (n + 1).$$

Therefore, from eq.(3), we get

$$0 \leq \lambda_i^{(i)} < 1. \tag{5}$$

In sum,

Commodity Exploitation Theorem. Given Assumptions 1 and 2, $\lambda_i^{(i)} < 1$ for any i .

The result $\lambda_i^{(i)} < 1$ in eq.(5) is claimed by some people to express the exploitation of commodity i . It should be noted that commodity i has been chosen in an arbitrary way, hence so long as the Hawkins-Simon condition is satisfied for the matrix $(I - \mathbb{A})$, all the commodities are ‘exploited’, including labour. (Eqs.(3) and (4) were obtained by Jeong (1982, 1984) and by Fujita (1991). These equations are now essential in the total flow analysis of Leontief models. See Szyrmer (1992) and Gallego and Lenzen (2005).)

We had better here explain a theorem by Fujita (2006) related to the above result.

Theorem. Let us assume the Hawkins-Simon condition for $(I - \mathbb{A}_{(i,i)})$. Then, the Hawkins-Simon condition for the extended matrix $(I - \mathbb{A})$ is equivalent to $\lambda_i^{(i)} < 1$.

This theorem can be proved as follows. Eq.(2) can be transformed to

$$\Lambda_{[i]}^{(i)}(I - \mathbb{A}) = (0, 0, \dots, 0, 1 - \lambda_i^{(i)}, 0, \dots, 0).$$

By the Cramer’s rule, we get, for the i -th variable,

$$1 = (1 - \lambda_i^{(i)}) \cdot \frac{|I - \mathbb{A}_{(i,i)}|}{|I - \mathbb{A}|} \text{ or } (1 - \lambda_i^{(i)}) = \frac{|I - \mathbb{A}|}{|I - \mathbb{A}_{(i,i)}|}.$$

Thus, $\lambda_i^{(i)} < 1$ if and only if $|I - \mathbb{A}| > 0$ because $|I - \mathbb{A}_{(i,i)}| > 0$. Thanks to a proposition by Georgescu-Roegen (1951), the condition $|I - \mathbb{A}| > 0$ is enough to guarantee the Hawkins-Simon condition for $(I - \mathbb{A})$.

The theorem due to Fujita tells us that when a productive system is enlarged to include one more commodity, the system can remain productive if and only if the value of the commodity in terms of that commodity is less than unity in the extended system.

3. A REFUTATION OF THE COMMODITY EXPLOITATION THEOREM

Eq.(5) in the previous section is mathematically nothing but an alternative expression of the Hawkins-Simon condition for the matrix $(I - \mathbb{A})$, and as such this is confined within the *physical* world, and has nothing to do with exploitation, as is warned by Okishio almost everywhere in Okishio (1977). Exploitation may be realized when exchanges at prices in the markets are practiced as the rules of the society orders. Eq.(5) tells us simply that the economic system is productive enough to produce a surplus in every sector. Who gets those surpluses is quite another problem. In order to discuss this problem, we need to introduce prices at which products are exchanged. Through exchanges of products, some obtain surpluses while others do not.

In our model, the price equations are written as

$$\begin{cases} p = (1 + r)(pA + \ell), \text{ and} \\ p \cdot c = 1, \end{cases}$$

where p is an n -row *wage-unit* price vector. In this section we have to make one more assumption about the *positiveness* of the money wage rate. That is, **Assumption (PW)**. The price equations above have a nonnegative solution. (Let us assume c has at least one positive entry, and call those commodities in c the *basic* consumption goods. Then, this assumption (PW) is guaranteed when the submatrix of $(A + c \cdot \ell)$ composed solely of basic consumption goods is indecomposable, and at least one process of basic consumption goods industries employs direct labour input. See Sraffa (1960).)

Actually, we do not have to solve the above equation. The important equation is $p \cdot c = 1$, and the fact that one unit of labour force is exchanged for the real wage consumption basket c .

Now, we consider the labour values of commodities, $\Lambda^{(n+1)}$, which is written as $\Lambda^{(\ell)}$ in this section. In a similar way, the index $(n + 1)$ is changed (ℓ) to stress that the variable concerned is related to labour. By definition, that is, by eq.(1), we have

$$\lambda_{\ell}^{(\ell)} = \Lambda_{(\ell)}^{(\ell)} \cdot c, \tag{6}$$

where

$$\Lambda_{(\ell)}^{(\ell)} \equiv (\lambda_1^{(\ell)}, \lambda_2^{(\ell)}, \dots, \lambda_n^{(\ell)}).$$

Eq.(6) tautologically means that one unit of labour is exchanged for the basket which has the same labour value. In this event, workers do not acquire any surplus. Thus,

Proposition 1. In the exchange between one unit of labour force and the workers' consumption basket, their labour values are equal.

Next let us consider the exchange between one unit of labour and the basket c in terms of the commodity i value. We assume that $c_i > 0$. The comparison

can then be expressed as follows:

$$\begin{aligned}\lambda_\ell^{(i)} &= \Lambda_{[i](\ell)}^{(i)} \cdot c && \text{from eq.(1)} \\ &> \Lambda_{(\ell)}^{(i)} \cdot c && \text{because } 1 > \lambda_i^{(i)} \text{ and } c_i > 0.\end{aligned}$$

This inequality demonstrates that when commodity i is included in the workers' consumption basket c , commodity i value of one unit of labour force is greater than commodity i value of the basket c . Hence,

Proposition 2. In the exchange between one unit of labour force and the workers' consumption basket, the commodity i value of labour is greater than that of basket provided commodity i is contained in the basket.

So, the above two mathematical propositions lead us to the following.

Corollary. The combined commodity basket c or its owners can obtain surpluses through exchanges with workers.

4. TWO NUMERICAL EXAMPLES

First, we take up an economy where there exists only one normal commodity, say, bread, and labour. The technological data are:

$$A = (0), \ell = (0.5), \text{ and } c = (1).$$

These gives

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ 0.5 & 0 \end{pmatrix} \text{ and } (I - \mathbb{A}) = \begin{pmatrix} 1 & -1 \\ -0.5 & 1 \end{pmatrix}.$$

It is easy to verify the matrix $(I - \mathbb{A})$ satisfies the Hawkins-Simon condition.

Now we can calculate

$$(I - \mathbb{A})^{-1} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

From eqs.(3) and (4), we get

$$\begin{aligned}\lambda_1^{(1)} &= \frac{2-1}{2} = 0.5 < 1, \lambda_\ell^{(1)} = \frac{2}{2} = 1; \\ \lambda_1^{(\ell)} &= \frac{1}{2} = 0.5, \lambda_\ell^{(\ell)} = \frac{2-1}{2} = 0.5 < 1.\end{aligned}$$

It is not difficult to compute the equilibrium wage-unit price of bread, and we have $p_1 = 1$ with the equilibrium rate of profit is 1. Thus, one unit of labour force is exchanged for the basket, i.e., one unit of bread equally in terms of labour values, both being 0.5. On the other hand, the exchange between one unit of labour force and one unit of bread in terms of bread values is unequal

since bread value of labour force is 1 while bread value of bread is 0.5, which verifies Proposition 2 in the preceding section.

Next, we consider an economy in which there are two normal commodities, bread and wine. The given data are:

$$A = \begin{pmatrix} 0.2 & 0.1 \\ 0.2 & 0.3 \end{pmatrix}, \ell = (1, 1), \text{ and } c = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix}.$$

These then yields

$$\mathbb{A} = \begin{pmatrix} 0.2 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.2 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } (I - \mathbb{A}) = \begin{pmatrix} 0.8 & -0.1 & -0.2 \\ -0.2 & 0.7 & -0.2 \\ -1 & -1 & 1 \end{pmatrix}.$$

The determinant of $(I - \mathbb{A})$ is 0.18, and thus positive, satisfying the Hawkins-Simon condition. The inverse of this matrix is computed as

$$(I - \mathbb{A})^{-1} = \begin{pmatrix} 25/9 & 5/3 & 8/9 \\ 20/9 & 10/3 & 10/9 \\ 5 & 5 & 3 \end{pmatrix}.$$

From eqs.(3) and (4), the values are:

$$\begin{aligned} \lambda_1^{(1)} &= \frac{25/9 - 1}{25/9} = \frac{16}{25} = 0.64 < 1, \lambda_2^{(1)} = \frac{5/3}{25/9} = 0.6, \lambda_\ell^{(1)} = \frac{8/9}{25/9} = 0.32; \\ \lambda_1^{(2)} &= \frac{20/9}{10/3} = \frac{20}{30} = 2/3, \lambda_2^{(2)} = \frac{10/3 - 1}{10/3} = 0.7 < 1, \lambda_\ell^{(2)} = \frac{10/9}{10/3} = 1/3; \\ \lambda_1^{(\ell)} &= \frac{5}{3}, \lambda_2^{(\ell)} = \frac{5}{3}, \lambda_\ell^{(\ell)} = \frac{3-1}{3} = 2/3 < 1. \end{aligned}$$

The equilibrium wage-unit prices are both $p_1 = p_2 = 2.5$, and the uniform rate of profit is 0.25. The comparison of values involved in the exchange of one unit of labour force and the consumption basket b are first made for bread value. Since

$$\lambda_\ell^{(1)} = 0.32 > \lambda_1^{(1)} \times 0.2 + \lambda_2^{(1)} \times 0.2 = 0.248,$$

the bread value of labour force is greater than that of basket. Second, in terms of wine value, we have

$$\lambda_\ell^{(2)} = 1/3 = 0.3 > \lambda_1^{(2)} \times 0.2 + \lambda_2^{(2)} \times 0.2 = 2/3 \times 0.2 + 0.7 \times 0.2 = 0.273$$

This inequality again confirms our Proposition 2.

5. CONCLUDING REMARKS

One of the important differences of labour from normal commodities is that labour is reproduced in the household sector, where no profit is earned, and one unit of labour is sold for the fixed consumption basket. The normal commodities are, on the other hand, produced in the production processes seeking for larger profits, and will have their prices distinct from the values in whichever commodity as the standard. Under various prices, each commodity can be exchanged for a different amount of other commodities. On the other hand, one unit of labour force is bound to be exchanged for one fixed consumption basket, whatever prices are going in the markets. To obtain Proposition 2, we do not have to calculate the equilibrium wage-unit prices. Or even the prevalence of equilibrium prices is not required. All we need to have is the relation $p \cdot c = 1$. Labour is the sole target of unequal exchange, i.e., exploitation, although surpluses are ‘born out’ collectively out of all the commodities. It should be agreed, though, that workers act as ‘midwives’ to realize these surpluses.

It is not difficult to generalize our results to a model of joint production. When the square material output coefficient matrix and the square input matrix are denoted by B and A , respectively, the extended net output matrix becomes

$$\mathbb{M} \equiv \begin{pmatrix} B - A & -c \\ -\ell & 1 \end{pmatrix}.$$

If $(B - A)$ is inverse-positive, i.e., $(B - A)^{-1} \geq 0$, then exactly the same arguments can be made so long as \mathbb{M} is also inverse-positive. Indeed \mathbb{M} is inverse-positive if $\Lambda \cdot c < 1$, thanks to the Banachiewicz identity related to the Schur complement (Banachiewicz (1937)): here $\Lambda \equiv \ell \cdot (B - A)^{-1}$. (See Fujimoto et al.(2003) for the inverse-positivity of $(B - A)$.)

The last remark is about the indecomposability of the matrices. We do not assume the indecomposability for the whole matrices of $(A+c \cdot \ell)$ or \mathbb{A} , which has been assumed in most of the literature related to the topic we have discussed in this note. In Section 3, while discussing the Commodity Exploitation Theorem, the matrix \mathbb{A} can even be decomposable, and the vectors c and ℓ are allowed to be zero! This fact also reveals that the theorem is totally disconnected from the social relationships. To discuss about exploitation, labour should be employed, and this at a positive wage rate.

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