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by Hiroaki Fujimoto and Masahito Irie [*]

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Abstract

More than eight decades ago, Frank P. Ramsey mathematically tackled a problem of optimal ad valorem tax rates in n-commodity markets, but just verbally derived a 'reciprocal elasticity rule' from an infinitesimal tax: That is, as stated in Ramsey (1927, p.56), the "infinitesimal ⋯ tax ad valorem on each commodity should be proportional to the sum of the reciprocals of its supply and demand elasticities." Since then, he and his followers have believed without any proof nor a closed-form solution that every rule stemmed from the infinitesimal tax "could be perfectly valid for a tax of 500% on whisky" as put in Ramsey (ibid., p.60). However, we would like to show in this paper that not only the limit of the $\frac{0}{0}$ form renowned as the l'Hôpital's rule but also Infinitesimal Calculus born in late 1960s reveals such a 'reciprocal elasticity rule' does not exist for good even with any infinitesimal tax. Instead of the 'reciprocal elasticity rule,' we employ another rule such as a tax revenue elasticity of a post-tax equilibrium quantity for an optimal indirect taxation, and explore some closed-form solutions that no one has yet shown in the literature. (*JEL* D61, H21)

It has been believed for more than eighty years that Frank P. Ramsey established a rule for an optimal indirect taxation, nowadays known as a 'reciprocal elasticity rule,' by which every optimal "infinitesimal ⋯ tax ad valorem on each commodity should be proportional to the sum of the reciprocals of its supply and demand elasticities" as mentioned in Ramsey (1927, p.56).

As social welfare, Ramsey takes an addition of u, "the *net* utility of producing and consuming (or saving) these quantities of commodities. This is usually regarded as the difference of two functions, one of which represents the utility of consuming, the other the disutility of producing (*ibid.*, p.48)." "*E. g.*, if $u = u_1 - u_2$ (consumers' utility − producers' disutility), $\dfrac{\partial u}{\partial x_r} = \dfrac{\partial u_1}{\partial x_r} - \dfrac{\partial u_2}{\partial x_r}$ = demand price of rth commodity − supply price = tax (*ibid.*, p.49)" as he denotes some of them by $u_1 = \displaystyle\int_0^{x_r} \phi_r(s_r)\, ds_r$, by $u_2 = \displaystyle\int_0^{x_r} f_r(s_r)\, ds_r$, and so by $\dfrac{\partial u}{\partial x_r} = \phi_r(x_r) - f_r(x_r) = \lambda_r(x_r) \equiv \lambda_r$ where x_r is a post-tax equilibrium quantity of a commodity, $\phi_r(x_r)$ is an inverse demand function, $f_r(x_r)$ is a supply function, and $\lambda_r = \lambda_r(x_r)$ is a tax, respectively.

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By maximizing its social welfare $\sum_{r=1}^{n} u = \sum_{r=1}^{n} \int_0^{x_r} \lambda_r(s_r)\, ds_r$,[1] Ramsey mathematically tackles the optimal indirect taxation problem with a few of parameters R, n, and so on,[2] in which a government raises her tax revenue $R \equiv \sum_{r=1}^{n} \lambda_r\, x_r$ from among n-commodity markets so as to answer a question of under a certain R, what ad valorem tax rates μ_r should be imposed on different commodities x_r for $r = 1, 2, \cdots, n$?[3] Assuming that "the utility is a non-homogeneous quadratic function of the x's, or that the λ's are linear (*ibid.*, p.52)," he manages to make the above 'reciprocal elasticity rule' from his "infinitesimal" tax $\lambda_r \approx 0$ or its sufficiently small tax revenue $R \approx 0$, and concludes without any parametric closed-form solution "that the results about "infinitesimal" taxes can only claim to be approximately true for small taxes, how small depending on data which are not obtainable. It is perfectly possible that a tax of 500% on whisky could for the present purpose be regarded as small (*ibid*, p.60)." According to Jerrold Marsden and Alan Weinstein (1985, p.73), however, it seems too early for him to conclude so because *Infinitesimal Calculus* was born in late 1960s.[4]

Besides, Ramsey (1927, p.58) discusses an implication of the 'reciprocal elasticity rule,' say a 'ratio rule' "of two commodities that should be taxed most which has the least elasticity of demand," and so that if it were completely inelastic, then "the whole of the revenue should be collected off it (*ibid*, pp.56-7)." To see it, recall based upon the 'reciprocal elasticity rule' that an optimal tax rate μ_r should be proportional to ($\propto$) a sum of reciprocals of both price elasticities of demand ρ_r ($\equiv \frac{-\phi_r}{\phi_r'\, x_r}$) and supply ε_r ($\equiv \frac{f_r}{f_r'\, x_r}$), or that $\mu_r \propto \frac{1}{\rho_r} + \frac{1}{\varepsilon_r}$ where ϕ_r' (< 0) is a derivative of the inverse demand function ϕ_r of the commodity x_r and f_r' (≥ 0) is that of supply function f_r, respectively. Suppose here that every market had a constant marginal cost of producing x_r or $f_r' = 0$, and so that ε_r always diverged to infinity or its reciprocal became zero. And, the 'reciprocal elasticity rule' turns to be as $\mu_r \propto \frac{1}{\rho_r}$ when a government needs her tax revenue by R. Although Ramsey does not tell us how its ρ_r differs from the other ρ_s nor any *ceteris paribus* condition on them, their 'ratio rule' of $\frac{\mu_r}{\mu_s}$ ($\propto \frac{\rho_s}{\rho_r}$) seems to tell us his implication that the less the elasticity ρ_r becomes, the more the rate μ_r should be levied. But, there is something wrong with this kind of negative relationship between them of $\mu_r \propto \frac{1}{\rho_r}$ whose derivative is negative as $\frac{d\,\mu_r}{d\,\rho_r}$ ($\propto -\frac{1}{\rho_r^2}$) < 0 because we have not only $\frac{d\,\mu_r}{d\,x_r} < 0$ as seen in footnote 3 but also $\frac{d\,\rho_r}{d\,x_r} \equiv \frac{\phi_r(\phi_r'' x_r + \phi_r') - (\phi_r')^2\, x_r}{(\phi_r'\, x_r)^2} \leq 0$ under an assumption of a well-behaved function ϕ_r with $\phi_r' < 0$, $\phi_r'' = 0$, or $\phi_r'' x_r + \phi_r' < 0$, and so that $\frac{d\,\rho_r}{d\,\mu_r} \geq 0$, which contradicts that $\frac{d\,\mu_r}{d\,\rho_r} < 0$.[5]

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[1] An idea of maximizing the *net* utility u can be interpreted as that of minimizing deadweight loss δ_r because their sum of $u + \delta_r = \int_0^{x_r} \lambda_r(s_r)\, ds_r + \int_{x_r}^{\bar{x}_r} \lambda_r(s_r)\, ds_r = \int_0^{\bar{x}_r} \lambda_r(s_r)\, ds_r \equiv \bar{u}_r$ should be fixed as an area over a domain of $\{\, x_r \,|\, 0 < x_r \leq \bar{x}_r \}$ where $\bar{x}_r$ is an initial equilibrium quantity of the commodity before tax such that its demand equals to the supply: *I.e.*, $\phi_r(\bar{x}_r) = f_r(\bar{x}_r)$.

[2] This belongs to the cost-benefit analysis typically used by a government to evaluate an intervention. A consumer's willingness to pay for x_r is measured by an area beneath ϕ_r as benefits whereas a producer's that to accept is one beneath f_r as costs.

[3] Without any explanation, he seems to change choice variables from the rates μ_r into the quantities x_r. So, our equations (4) and (5) show us the explanation of strict monotone decreasing as $\frac{d\,\mu_r}{d\,x_r} < 0$ in the domain of $\{\, x_r \,|\, 0 < x_r \leq \bar{x}_r \}$ in footnote 1.

[4] Also, see the last paragraph in appendix C of this paper and Jerome H. Keisler (2002, pp.33-76).

[5] This contradiction is up to equation (11) of Ramsey (1927, p.56) as $\mu_r = \frac{\theta(1/\varepsilon_r + 1/\rho_r)}{1 - \theta/\rho_r}$ where we underline to make it different from ours and θ is a multiplier. As proved in appendix C, his equation (11) should have been evaluated as our equation (C11).

By using the total-differential approach, Ramsey (1927, pp.48-9) obtains a set of equations for the first-order condition,[6] but he does not show us a rule like the Hermann Heinrich Gossen's Second Law.[7] According to Francis Y. Edgeworth (1925, pp.100-22), there are some rules on 'sacrifice' to a certain direct tax-payment such as 'equal-sacrifice,' 'proportional-sacrifice,' 'equal-marginal-sacrifice,' *etc.* So, Ramsey should have provided us with at least an equal-marginal-sacrifice rule of an optimal indirect taxation.

Therefore, we cannot help having two main purposes here: One is to reexamine those Ramsey tax rules such as the 'reciprocal elasticity rule,' its 'ratio rule,' and the 'equal-marginal-sacrifice rule;' the other is to find out some parametric closed-form solutions that no one has yet shown before in the literature. To engage in them, the rest of our paper is organized as follows. First of all, section I presents the limit of the $\frac{0}{0}$ form renowned as the Guillaume de l'Hôpital's rule discloses that both of the 'reciprocal elasticity rule' and its 'ratio rule' do not work: That is, none of infinitesimal tax rates can be proportional to a sum of reciprocals of price elasticities of demand and supply at all even with a horizontal supply function because the sum is always too large to have the same order as infinitesimal candidates in the $\frac{0}{0}$ form. Second, section II shows us a few of new rules of indirect taxation: For example, we take a tax revenue elasticity of equilibrium quantity as an equal-elasticity-sacrifice rule, and explore several parametric examples in order to seek some closed-form solutions. Finally, section III concludes this paper.

We put all proofs in appendix A; in the meantime, appendixes B and C are also prepared for the second-order condition and for a review of Ramsey (1927), respectively.

I. The Ramsey's Indirect Taxation

A. *The Model based upon Ramsey's Part III*

Just as Frank P. Ramsey (1927, p.55-8), consider an optimization problem where a government imposes an ad valorem tax rate μ_r on n commodities with a running index r as $r = 1, 2, \cdots$, and n in order to collect her tax revenue by an amount of R. Let p_r and x_r be a price and a quantity of the r-th commodity, respectively, then assume in its commodity market that she faced not only an inverse demand function of $p_r = \phi_r(x_r)$ with a negative slope of $\phi_r'(x_r) \equiv \dfrac{d\,\phi_r(x_r)}{d\,x_r} < 0$ but also a supply function of $p_r = f_r(x_r)$ with a

[6] First of all, he has not only that $\sum_{r=1}^{n} \lambda_r\, dx_r = 0$, say equation (♣) but also that $\sum_{r=1}^{n}(\lambda_r + \lambda_r'\, x_r)dx_r = 0$, say equation (♢). Even though he should have had from the latter equation (♢) that $dx_n = -\sum_{r=1}^{n-1} \frac{\lambda_r+\lambda_r'\, x_r}{\lambda_n+\lambda_n'\, x_n} dx_r$, say equation (♡), he has next had that $\sum_{r=1}^{n} \lambda_r'\, x_r dx_r = 0$, say equation ($\underline{♢}$) by subtracting equation (♣) from equation (♢). Third, instead of equation (♡), he has from equation ($\underline{♢}$) that $dx_n = -\sum_{r=1}^{n-1} \frac{\lambda_r'\, x_r}{\lambda_n'\, x_n} dx_r$, say equation ($\underline{♡}$). Fourth, substituting equation ($\underline{♡}$) into equation (♣) yields to him that $\sum_{r=1}^{n-1}(\lambda_r - \lambda_n \frac{\lambda_r'\, x_r}{\lambda_n'\, x_n})dx_r = 0$, say equation ($\underline{♠}$), from which we must have that $\lambda_r = \lambda_n \frac{\lambda_r'\, x_r}{\lambda_n'\, x_n}$ for $r = 1, 2, \cdots, n-1$ owing to $dx_r \neq 0$. At last, it is nothing more than equation (3) of Ramsey (1927, p.55) where we underline to make it different from ours, or $\frac{\lambda_1}{\lambda_1'\, x_1} = \frac{\lambda_2}{\lambda_2'\, x_2} = \cdots = \frac{\lambda_n}{\lambda_n'\, x_n} = -\theta$. So, it is now easy to confirm as mentioned in footnotes 4 and 5 that the multiplier θ is such a function of a choice variable x_r that we had better not take the limit of $\mu_r = \frac{\theta(1/\varepsilon_r+1/\rho_r)}{1-\theta/\rho_r}$ by θ only.

[7] This Law is a rule such that in an equilibrium, specially at a stationary point of the first-order condition, a consumer should allocate expenditures so that the ratio of marginal utility to price equals across all commodities as well as to a multiplier. As usual as stated in footnote 6, by substituting equation (♡) into equation (♣) we have that $\sum_{r=1}^{n-1}(\lambda_r - \lambda_n \frac{\lambda_r+\lambda_r'\, x_r}{\lambda_n+\lambda_n'\, x_n})dx_r = 0$, say equation (♠), from which we can have that $\lambda_r = \lambda_n \frac{\lambda_r+\lambda_r'\, x_r}{\lambda_n+\lambda_n'\, x_n}$ for $r = 1, 2, \cdots, n-1$. See footnote 12 for further discussion.

non-negative slope of $f_r'(x_r) \equiv \dfrac{d\, f_r(x_r)}{d\, x_r} \geq 0$, and so that she could define a vertical length λ_r between two functions independently on x_r as

(1) $\quad \lambda_r = \lambda_r(x_r) \equiv \phi_r(x_r) - f_r(x_r) \geq 0$

in a domain of

(2) $\quad 0 < x_r \leq \bar{x}_r$

where $\bar{x}_r$ is an initial equilibrium quantity such that $\phi_r(\bar{x}_r) = f_r(\bar{x}_r)$ when there is no taxation on the r-th commodity with the tax rate of $\mu_r = 0$.

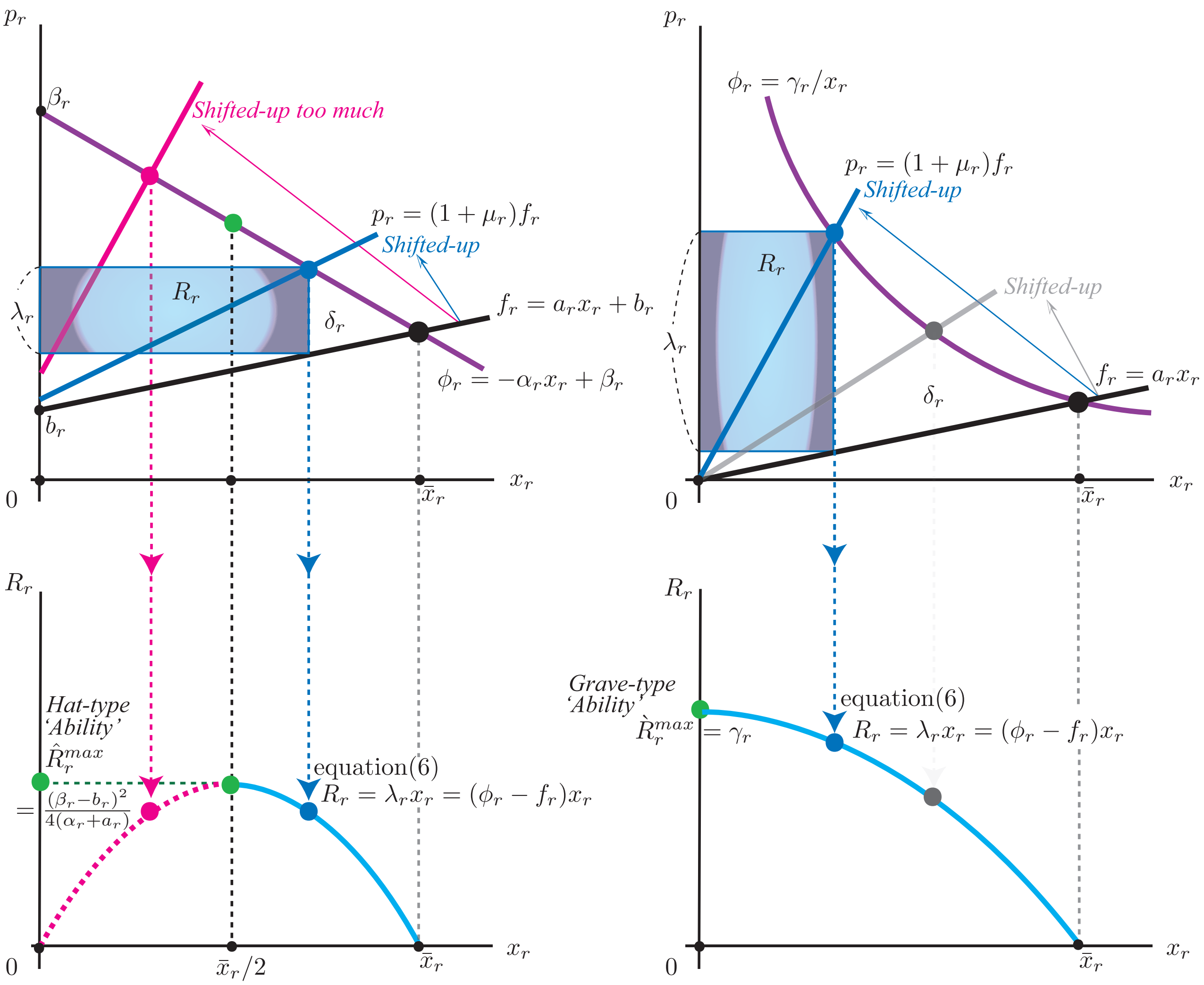

Figure 1: Two Types of Rectangular Areas with the Maximum Amount of Tax R_r^{max} Payable

It is known that if some tax rate $\mu_r > 0$ is imposed, geometrically speaking, the supply function $p_r = f_r(x_r)$ shifts up to $p_r = (1+\mu_r)f_r(x_r)$ along the inverse demand function $p_r = \phi_r(x_r)$ as seen in figure 1, so that a new equilibrium quantity x_r after tax should satisfy an equation of $\phi_r(x_r) = (1+\mu_r)f(x_r)$. Thus, equation (1) or a difference between two functions λ_r becomes

$$(3) \quad \lambda_r = \phi_r(x_r) - f_r(x_r) = \mu_r f_r(x_r),$$

then the ad valorem tax rate μ_r can be expressed as

$$(4) \quad \mu_r = \frac{\phi_r(x_r) - f_r(x_r)}{f_r(x_r)} = \frac{\lambda_r(x_r)}{f_r(x_r)} = \frac{\lambda_r}{f_r} \geq 0$$

in terms of the quantity x_r. So, it is easy to see by taking the derivative of equation (4) with respect to x_r in the domain of equation (2) as

$$(5) \quad \mu_r' \equiv \frac{d\,\mu_r}{d\,x_r} = \frac{\phi_r'(x_r) f_r(x_r) - \phi_r(x_r) f_r'(x_r)}{\{f_r(x_r)\}^2} < 0$$

that the tax rate μ_r is monotonously decreasing with respect to x_r owing to the signs of slopes $\phi_r' < 0$ and $f_r' \geq 0$, and so that the government can treat the quantity x_r as her choice variable instead of the rate μ_r.[8]

It is also known that if the tax rate $\mu_r > 0$ is levied, then a rectangular area, namely, a product of the length λ_r in equation (1) or (3) and a width of the quantity x_r makes a tax revenue of

$$(6) \quad R_r \equiv \lambda_r\, x_r = \{\phi_r(x_r) - f_r(x_r)\}\, x_r \geq 0.$$

As shown in figure 1, there are two types of rectangular areas with the maximum amount of tax denoted by R_r^{max} potentially payable to the government by the r-th commodity market, or market's 'ability.'[9] To find out the 'ability' of R_r^{max} in the r-th market, we usually attempt the following two ways: The first way is to take the derivative of equation (6) with respect to x_r and set it equal to zero if we can do it so as

$$(7) \quad \frac{d\,R_r}{d\,x_r} \equiv \lambda_r'\, x_r + \lambda_r = 0$$

where λ_r' is the derivative of equation (1) or (3) with respect to x_r of

$$(8) \quad \lambda_r' \equiv \frac{d\,\lambda_r(x_r)}{d\,x_r} = \phi_r'(x_r) - f_r'(x_r) < 0 \quad \text{for } r = 1, 2, \cdots, n$$

anywhere in the domain of equation (2) or $0 < x_r \leq \bar{x}_r$ due to the signs of slopes $\phi_r' < 0$ and $f_r' \geq 0$. Because of equation (7), we acquire the maximum amount of $R_r^{max} = \hat{R}_r^{max}$ at $x_r = x_r^{max}$ such that $\dfrac{-\lambda_r}{\lambda_r'\, x_r} = 1$ in its domain $(0 < x_r \leq \bar{x}_r)$. For example, the government may levy the maximum 'ability' of $\hat{R}_r^{max} \equiv \dfrac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$, a Hat-type in figure 1 as her revenue at $x_r^{max} \equiv \dfrac{\beta_r - b_r}{2\,(\alpha_r + a_r)}$ on an affine market where she faces not merely an inverse demand function $\phi_r \equiv -\alpha_r\, x_r + \beta_r$ but also a supply function $f_r \equiv a_r\, x_r + b_r$ with parameters

[8] In the case of a unit tax υ_r, we can replace equations (1) or (3), (4), and (5) by $\lambda_r = \phi_r(x_r) - f_r(x_r) = \upsilon_r \geq 0$, $\upsilon_r = \phi_r(x_r) - f_r(x_r) \geq 0$, and $\upsilon_r' \equiv \frac{d\,\upsilon_r}{d\,x_r} = \phi_r'(x_r) - f_r'(x_r) < 0$, respectively since the supply function $p_r = f_r(x_r)$ shifts up parallel to $p_r = f_r(x_r) + \upsilon_r$ by the unit tax υ_r along the inverse demand function $p_r = \phi_r(x_r)$.

[9] Since we are able to find a word of 'ability' in Adam Smith as individuals' income tax payments to "contribute in proportion to their respective abilities," a basic idea here is a replacement of an individual on a direct tax by a market on an indirect one.

$\alpha_r > 0$, $\beta_r > b_r \geq 0$, and $a_r \geq 0$ but $a_r = b_r \neq 0$ simultaneously; on the other hand, the second one is to take the limit of equation (6) as x_r goes to the origin 0 from the right hand side denoted by $^+$. That is, when she meets in a market that

$$(9) \quad \frac{d\,R_r}{d\,x_r} \equiv \lambda'_r\,x_r + \lambda_r < 0$$

everywhere in the domain of equation (2) or $0 < x_r \leq \bar{x}_r$. From equation (6), in this case, she attains its maximum 'ability' of a tax revenue $R_r^{max} = \grave{R}_r^{max}$, or

$$(10) \quad \lim_{x_r \to 0^+} R_r \to R_r^{max} = \grave{R}_r^{max} > 0$$

as the length λ_r in equation (1) or (3) reaches its maximum at $x_r = 0$ from equation (8). As seen on the right hand side in figure 1, *e.g.*, a demand function of $\phi_r \equiv \dfrac{\gamma_r}{x_r}$, $\gamma_r > 0$ yields to the government Grave-type 'ability' of its maximum $\grave{R}_r^{max} \equiv \gamma_r$ because of the limit of $\lim_{x_r \to 0^+} R_r \to \dfrac{\gamma_r}{x_r}\,x_r - f_r(0)\,0 = \gamma_r$.

In addition to equations (7) and (9) of

$$(11) \quad \frac{d\,R_r}{d\,x_r} \equiv \lambda'_r\,x_r + \lambda_r \leq 0,$$

not only equation (6), often a mountain shape as the Arther Laffer curve, can give us at most one summit of the maximum revenue or the Hat-type 'ability' $\hat{R}_r^{max}$ in the domain of equation (2) as $0 < x_r \leq \bar{x}_r$, but other mountains of equation (6) hidden in the second quadrant ($x_r < 0$) should possess a quasi-concavity as

$$(12) \quad \frac{d^2 R_r}{d\,x_r^2} \equiv \lambda''_r\,x_r + 2\,\lambda'_r \leq 0.$$

So, equation (12) has an important role, *e.g.*, for the second-order condition as seen in appendix B. It is worth noticing here that for a tax problem with an amount $R_r^{\exists}$ less than the 'ability' R_r^{max} in equation (6): That is, for its $R_r^{max} > R_r^{\exists} \geq 0$, equation (9) prevails to equation (7) or

$$(13) \quad \frac{d\,R_r}{d\,x_r} \equiv \lambda'_r\,x_r + \lambda_r < 0$$

at $x_r = x_r^{R_r^{\exists}}$ in the domain of equation (2) as it always happens somewhere on the right of the maximum in equation (6) as $0 \leq x_r^{max} < x_r^{R_r^{\exists}} = x_r \leq \bar{x}_r$.

The government is now able to take the quantity x_r as a choice variable due to equation (5) in order to maximize an objective function $\mathcal{U}$ of social welfare, a sum of market-surplus $ms_r \equiv \int_0^{x_r} \lambda_r(s_r)\,ds_r$ subject to a constraint, the constant tax revenue R: *I.e.*, she $\underset{x_1, x_2, \cdots, x_n}{\text{maximizes}}\ \mathcal{U} \equiv \sum_{r=1}^{n} ms_r$ subject to $R = \sum_{r=1}^{n} R_r$; or she

$$(14) \quad \underset{x_1, x_2, \cdots, x_n, \kappa}{\text{maximizes}}\ \mathcal{L} \equiv \sum_{r=1}^{n} \int_0^{x_r} \lambda_r\,ds_r + \kappa\,(R - \sum_{r=1}^{n} \lambda_r\,x_r)$$

where $\mathcal{L}$ is a Lagrange-function with a multiplier κ.[10] Taking partial derivatives of equation (14) with respect to choice variables, $x_1, x_2, \cdots, x_n$, and κ yields to us a set of the following first-order condition,[11]

$$\frac{\partial \mathcal{L}}{\partial x_r} = \lambda_r - \kappa(\lambda_r + \lambda_r' x_r) = 0 \quad \text{for } r = 1, 2, \cdots, n;$$
$$\frac{\partial \mathcal{L}}{\partial \kappa} = R - \sum_{r=1}^{n} \lambda_r x_r = 0.$$

To summarise the above $(n+1)$ equations, let us replace the multiplier κ by his multiplier $-K$ for a while, as seen in Ramsey (1927, p.50), then we have them as follows:[12]

$$(15) \quad K = K_r = K_r(x_r) \equiv -\frac{\lambda_r}{\lambda_r + \lambda_r' x_r} = -\kappa \qquad \text{for } r = 1, 2, \cdots, n;$$

$$(16) \quad R = \sum_{r=1}^{n} R_r = \sum_{r=1}^{n} \lambda_r x_r.$$

B. *Proportionality within* $0 < x_r < \bar{x}_r$

Since Ramsey (1927, p.60) insists on some linearity or proportionality about a tax of 500% from an infinitesimal tax, we look over here if choice variables including the multiplier as well as tax functions go through the origin 0. It is obvious in an open set of $0 < x_r < \bar{x}_r$ came from equation (2) that his multiplier $K_r = K_r(x_r)$ in equation (15) cannot be proportional to the others for all $r = 1, 2, \cdots, n$ since the limit of $K_r(x_r)$ never converges to zero as x_r approaches zero from the right (0^+) :

$$(17) \quad \lim_{x_r \to 0^+} K_r(x_r) = -\frac{\lambda_r}{\lambda_r + \lambda_r' \times 0} \begin{cases} = -1 \neq 0 & \text{if } \lambda_r' \not\to -\infty \text{ simultaneously;} \\ \to +\infty \neq 0 & \text{if } \lambda_r' \to -\infty \text{ simultaneously.} \end{cases}$$

Mathematically speaking, equation (17) means that they never go through the origin 0, and so that nobody can imagine any linear relationship among them (those multipliers $K_1, K_2, \cdots, K_n$ and their quantities $x_1, x_2, \cdots, x_n$). Accordingly, nobody should have put them into straight lines going through the origin 0 even approximations $(\approx)$ of them like

$$(18) \quad K = K_r = K_r(x_r) = -\frac{\lambda_r}{\lambda_r + \lambda_r' x_r} \not\approx c_r x_r$$

with a constant coefficient c_r everywhere in the set of $0 < x_r < \bar{x}_r$ for all $r = 1, 2, \cdots, n$. In fact, moreover, an identical value of the multiplier K, say $K = K^*$ to every his multiplier K_r, does not always provide us with a linearity among them because the function of $K^* = K_r(x_r)$ does not always go through the origin 0

[10] Owing to a deadweight loss δ_r as seen in figure 1, she maximizes$_{x_1, x_2, \cdots, x_n, \kappa}$ $\mathcal{L} \equiv -\sum_{r=1}^{n} \delta_r + \kappa \, (R - \sum_{r=1}^{n} \lambda_r x_r)$ with the fact of $u_r = \bar{u}_r - \delta_r$ as discussed in footnote 1 where $u_r \equiv \int_0^{x_r} \lambda_r(s_r)\, ds_r$, $\delta_r \equiv \int_{x_r}^{\bar{x}_r} \lambda_r(s_r)\, ds_r$, and $\bar{u}_r \equiv \int_0^{\bar{x}_r} \lambda_r(s_r)\, ds_r$.

[11] See appendix B for the second-order condition in our program of equation (14): That is, a stationary value of the Lagrange-function $\mathcal{L}$ from parametric or numerical solutions, say $\bar{\mathcal{L}}$ needs to be tested against a second-order condition called a bordered Hessian. For this Hessian, *e.g.*, see Alpha C. Chiang (1984, p.375, pp.379-87), who gives us a word of caution on the multiplier $\kappa = -K$, which is also a variable to be solved in terms of parameters initially given, reflecting that $\kappa = \frac{d\mathcal{L}}{dR}$ here.

[12] As addressed in footnote 7, the total-differential approach should have yielded to us that $\lambda_r = \lambda_n \frac{\lambda_r + \lambda_r' x_r}{\lambda_n + \lambda_n' x_n}$ for $r = 1, 2, \cdots, n-1$, and so that $\frac{\lambda_1}{\lambda_1 + \lambda_1' x_1} = \frac{\lambda_2}{\lambda_2 + \lambda_2' x_2} = \cdots = \frac{\lambda_n}{\lambda_n + \lambda_n' x_n} = \kappa$ (say). It is nothing but our equation (15).

at $x_r = 0$. Furthermore, we can apply those rules used in equations (17) and (18) to vanishing any possibility of proportionality or linearity. It is easy to see almost everywhere in the set of $0 < x_r < \bar{x}_r$ except in the neighborhood of $\bar{x}_r$ that each indirect tax is not proportional to any function of x_r, say $\sigma_r = \sigma_r(x_r)$, *i.e.*, as $x_r \to 0^+$, so that not merely the limit of an ad valorem tax rate μ_r but also that of a unit tax υ_r never converges to zero but a certain positive number this time or both of them diverge to positive infinity due to its monotone decreasing with respect to x_r as seen in equations (4), (5), and footnote 8:

$$(19)\quad \lim_{x_r \to 0^+} \mu_r = \lim_{x_r \to 0^+} \mu_r(x_r) = \frac{\phi_r(0) - f_r(0)}{f_r(0)} > 0;$$

$$(20)\quad \lim_{x_r \to 0^+} \upsilon_r = \lim_{x_r \to 0^+} \upsilon_r(x_r) = \phi_r(0) - f_r(0) > 0,$$

which means any indirect tax never passes through the origin 0 at all at $x_r = 0$.

C. *Proportionality in the Neighborhood of* $x_r = \bar{x}_r$

In the meantime, it is obvious at every point close enough to $x_r = \bar{x}_r$ in the set of $0 < x_r < \bar{x}_r$ for all $r = 1, 2, \cdots, n$ that infinitesimal multipliers $K_r(x_r)$ in equation (15) can be proportional each other because its value of $K_r(x_r)$ must be small as $K_r(x_r) \approx 0$ in taking the limit as x_r tends to the initial one of $\bar{x}_r$ from the left hand side denoted by $^-$. As it can be seen in a market at $x_r = \bar{x}_r$ before tax that we have $\phi_r(\bar{x}_r) = f_r(\bar{x}_r)$ from the demand = supply condition or $\phi_r(\bar{x}_r) - f_r(\bar{x}_r) = 0$, the following left-hand limit shows us

$$(21)\quad \lim_{x_r \to \bar{x}_r^-} K_r(x_r) = -\frac{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)}{\phi_r(\bar{x}_r) - f_r(\bar{x}_r) + \lambda_r'(\bar{x}_r) \times \bar{x}_r} = 0$$

with a non-zero product of equation (8) or λ_r' (< 0) and $\bar{x}_r$ (> 0). It is easy for us to seek another candidate of infinitesimal function of x_r around $\bar{x}_r$: For instance, equations (1) or (3), (4), (6), and footnote 8 give us

$$(22)\quad \lim_{x_r \to \bar{x}_r^-} \lambda_r = \lim_{x_r \to \bar{x}_r^-} \lambda_r(x_r) = \phi_r(\bar{x}_r) - f_r(\bar{x}_r) = 0;$$

$$(23)\quad \lim_{x_r \to \bar{x}_r^-} \mu_r = \lim_{x_r \to \bar{x}_r^-} \mu_r(x_r) = \frac{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)}{f_r(\bar{x}_r)} = 0;$$

$$(24)\quad \lim_{x_r \to \bar{x}_r^-} R_r = \lim_{x_r \to \bar{x}_r^-} R_r(x_r) = \{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)\}\, \bar{x}_r = 0;$$

$$(25)\quad \lim_{x_r \to \bar{x}_r^-} \upsilon_r = \lim_{x_r \to \bar{x}_r^-} \upsilon_r(x_r) = \phi_r(\bar{x}_r) - f_r(\bar{x}_r) = 0.$$

So, *e.g.*, a revenue R_r is proportional to $(\propto)$ an ad valorem tax rate μ_r as $R_r \propto \mu_r$ with a straight line of $R_r = c_r\, \mu_r$ nearby at $x_r \leq \bar{x}_r$ for an infinitesimal $\mu_r \geq 0$ where a constant $c_r \equiv \phi_r(\bar{x}_r)\, \bar{x}_r$ is given as follows:[13]

$$(26)\quad \frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{R_r}{\mu_r} = \frac{\{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)\}\, \bar{x}_r}{\{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)\}/f_r(\bar{x}_r)} = f_r(\bar{x}_r)\, \bar{x}_r \equiv c_r,$$

which also supplies us with the famous Guillaume de l'Hôpital's rule of

$$(27)\quad \lim_{x_r \to \bar{x}_r^-} \frac{R_r}{\mu_r} = \lim_{x_r \to \bar{x}_r^-} \frac{\{R_r(x_r) - R_r(\bar{x}_r)\}/(x_r - \bar{x}_r)}{\{\mu_r(x_r) - \mu_r(\bar{x}_r)\}/(x_r - \bar{x}_r)} = \frac{R_r'(\bar{x}_r)}{\mu_r'(\bar{x}_r)}.$$

[13] For a sufficiently small unit tax $\upsilon_r \geq 0$, a revenue R_r is proportional to its υ_r as $R_r \propto \upsilon_r$ at $x_r \leq \bar{x}_r$ or $R_r = c_r\, \upsilon_r$ with a constant coefficient c_r from the left-hand limit of $\frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{R_r}{\upsilon_r} = \bar{x}_r \equiv c_r$. See Chiang (1984, pp.428-30) for the $\frac{0}{0}$ form.

Consequently, all we have to do is to show whether or not a function of x_r, say $\sigma_r = \sigma_r(x_r)$ is convergent to zero as x_r approaches $\bar{x}_r$ in order that an infinitesimal σ_r gets proportional to sufficiently small indirect taxes, μ_r and υ_r for all $r = 1, 2, \cdots, n$. Otherwise, any $\sigma_r(x_r)$ is dominated by the limit like equations (17), (19), and (20) since every real number x_r in the set of $0 < x_r < \bar{x}_r$ is now way far from the origin 0. Due to equations (19) and (20), needless to say, none of $\sigma_r(x_r)$ should be proportional to an ad valorem tax rate $\mu_r(x_r)$ nor a unit tax $\upsilon_r(x_r)$ almost everywhere except in the neighborhood of $x_r = \bar{x}_r$.

In this sense, denote a sum of reciprocals of a price elasticity of demand and that of supply by $\sigma_r = \sigma_r(x_r)$ for $r = 1, 2, \cdots, n$, namely,

$$(28)\quad \sigma_r \equiv \frac{1}{\rho_r} + \frac{1}{\varepsilon_r} = -\frac{\phi_r'(x_r)\, x_r}{\phi_r(x_r)} + \frac{f_r'(x_r)\, x_r}{f_r(x_r)}$$

where ρ_r is a price elasticity of demand or $\rho_r \equiv -\dfrac{\phi_r(x_r)}{\phi_r'(x_r)\, x_r}$ and ε_r is that of supply or $\varepsilon_r \equiv \dfrac{f_r(x_r)}{f_r'(x_r)\, x_r}$, then we have the following proposition:[14]

PROPOSITION 1: *The Ramsey's 'reciprocal elasticity rule,' which claims anywhere even close enough to $x_r = \bar{x}_r$ that an ad valorem tax rate μ_r could be proportional to ($\propto$) the sum of reciprocals of price elasticities of demand and supply $\sigma_r(x_r) = \dfrac{1}{\rho_r} + \dfrac{1}{\varepsilon_r}$, is not true: I.e., $\mu_r \not\propto \sigma_r(x_r)$ even for a sufficiently small μ_r; but $\mu_r \propto K_r$ in the neighborhood of $x_r \leq \bar{x}_r$ through a linear relationship $\mu_r = \sigma_r(\bar{x}_r)\, K_r$ for infinitesimal tax rates $\mu_r \geq 0$ from equation (4) and his multipliers K_r from equation (15) for all $r = 1, 2, \cdots, n$.*

Proof. See appendix A for this proof. □

We can see in appendix A that even with a horizontal supply function, a case of slope-zero or $f_r'(x_r) = 0 = \dfrac{1}{\varepsilon_r}$, our proposition 1 is robust, and from equation (C14) of appendix C that proportionality with exactly the same coefficients $\sigma_r(\bar{x}_r)$ as $\mu_r = \sigma_r(\bar{x}_r)\, \theta_r$ hold among these rates μ_r and length elasticities θ_r in equation (C3) nearby at $x_r \leq \bar{x}_r$ for all $r = 1, 2, \cdots, n$.

Anyway, equation (A4) gives us another $\dfrac{0}{0}$ form like equation (26) as well as the l'Hôpital's rule like equation (27) as

$$(29)\quad \frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{\mu_r}{K_r} = \lim_{x_r \to \bar{x}_r^-} \frac{\{\mu_r(x_r) - \mu_r(\bar{x}_r)\}/(x_r - \bar{x}_r)}{\{K_r(x_r) - K_r(\bar{x}_r)\}/(x_r - \bar{x}_r)} = \frac{\mu_r'(\bar{x}_r)}{K_r'(\bar{x}_r)} = c_r$$

owing to $\mu_r(\bar{x}_r) = K_r(\bar{x}_r) = 0$ from equations (23) and (21). Equation (29) converges to the constant value c_r as shown in equations (A3) and (A4) as

$$(\text{A3})\quad c_r = \lim_{x_r \to \bar{x}_r^-} \sigma_r(x_r) = \sigma_r(\bar{x}_r) = -\frac{\lambda_r'(\bar{x}_r)\, \bar{x}_r}{f_r(\bar{x}_r)} = \frac{1}{\rho_r(\bar{x}_r)} + \frac{1}{\varepsilon_r(\bar{x}_r)},$$

[14] See appendix C for a review about a few equations in Ramsey (1927).

so that by using the unit-less quantity q_r in equation (A5), one may estimate a rate $\tilde{\mu}_r$ approximately with an infinitesimal multiplier K_r as

$$(30) \quad \tilde{\mu}_r = \tilde{\mu}_r(K_r) \equiv \sigma_r(\bar{x}_r)\, K_r(\bar{x}_r\, q_r)$$

at $q_r \equiv \dfrac{x_r}{\bar{x}_r} \leq 1$ when x_r is close enough to $\bar{x}_r$ or equal to it. Even for any infinitesimal μ_r, thus, we have $\mu_r \not\propto \sigma_r$, but $\mu_r \propto K_r$ instead: *I.e.*, any ad valorem tax rate μ_r is never proportional to σ_r, the sum of reciprocals of price elasticities of demand and supply; but to his multiplier K_r in the neighborhood of $\bar{x}_r$ to retain the nil term of $[\{\lambda_r(x_r)\}] \approx 0$ in equation (A4). Neither is a unit tax υ_r.[15]

COROLLARY 1: *The Ramsey's tax 'ratio rule,' which claims even nearby at $x_r = \bar{x}_r$ that a ratio of ad valorem tax rates $\dfrac{\mu_i}{\mu_j}$ could be proportional to ($\propto$) that of the sum of reciprocals of price elasticities $\dfrac{\sigma_i(x_i)}{\sigma_j(x_j)}$, is not true, either: i. e., $\dfrac{\mu_i}{\mu_j} \not\propto \dfrac{\sigma_i(x_i)}{\sigma_j(x_j)}$ at all; on the contrary, $\dfrac{\mu_i}{\mu_j} \propto \dfrac{K_i(x_i)}{K_j(x_j)}$ such as $\dfrac{\mu_i(x_i)}{\mu_j(x_j)} = \dfrac{\sigma_i(\bar{x}_i)}{\sigma_j(\bar{x}_j)}\,\dfrac{K_i(x_i)}{K_j(x_j)}$ for $x_r \leq \bar{x}_r$ but very close to $\bar{x}_r$ corresponding to infinitesimal tax rates of $\mu_r \geq 0$ from equation (4) over his multipliers K_r from equation (15) for all elements i, j, and $r \in \{s \mid s = 1, 2, \cdots, n\}$.*

Proof. It is obvious from proposition 1, but see appendix A for this. □

It is also easy for us to show for all elements i, j, and $r \in \{s \mid s = 1, 2, \cdots, n\}$ from footnote 15 that we can appreciate the following limit of a ratio of

$$(31) \quad \lim_{q_r \to 1^-} \frac{\upsilon_i/K_i}{\upsilon_j/K_j} = \lim_{q_r \to 1^-} \frac{f_i(\bar{x}_i\, q_i)}{f_j(\bar{x}_j\, q_j)}\,\frac{\sigma_i(\bar{x}_i\, q_i)}{\sigma_j(\bar{x}_j\, q_j)} = \frac{f_i(\bar{x}_i)}{f_j(\bar{x}_j)}\,\frac{\sigma_i(\bar{x}_i)}{\sigma_j(\bar{x}_j)} \equiv c^{\upsilon}_{ij},$$

as the unit-less q_r tends to unity from the left in equation (A5), $0 < q_r \equiv \dfrac{x_r}{\bar{x}_r} \leq 1$. So, we are able to see nearly at $q_r = 1$ with negligible length of equation (1) or (3) of $[\{\lambda_r(\bar{x}_r\, q_r)\}] \approx 0$ that a ratio of unit taxes $\dfrac{\upsilon_i}{\upsilon_j} \not\propto \dfrac{\sigma_i(\bar{x}_i\, q_i)}{\sigma_j(\bar{x}_j\, q_j)}$, a ratio of the sums from equation (28); but the ratio of them $\dfrac{\upsilon_i}{\upsilon_j} \propto \dfrac{K_i(\bar{x}_i\, q_i)}{K_j(\bar{x}_j\, q_j)}$, a ratio of his multipliers with a linear relation $\dfrac{\upsilon_i(\bar{x}_i\, q_i)}{\upsilon_j(\bar{x}_j\, q_j)} = c^{\upsilon}_{ij}\,\dfrac{K_i(\bar{x}_i\, q_i)}{K_j(\bar{x}_j\, q_j)}$ where c^{υ}_{ij} is given by equation (31) as a constant coefficient for infinitesimal unit tax $\upsilon_r \geq 0$ upon his multipliers K_r very nearby at $q_r < 1$.

One may wonder in corollary 1 that equations (A6) and (31) could have implicated in a ratio of optimal indirect taxes as $\dfrac{\mu^*_i}{\mu^*_j}$ or $\dfrac{\upsilon^*_i}{\upsilon^*_j}$ because their numerators' K_i and denominators' K_j are seemingly canceled out by a common infinitesimal his multiplier $K^* = K_r(x^*_r) \neq 0$ given somewhere at $x_r = x^*_r < \bar{x}_r$ close enough to $\bar{x}_r$ or at $q_r = q^*_r < 1$, but they have implicated in that of rates of change of them as $\dfrac{d\,\mu_i}{d\,\mu_j}$ or $\dfrac{d\,\upsilon_i}{d\,\upsilon_j}$ because those infinitesimal variables are usually, mathematically speaking, denoted by $d\,K_r$, $d\,\mu_r$, $d\,\upsilon_r$, and so on for

[15]The $\frac{0}{0}$ form of equations (25) to (21) or $\lim_{x_r \to \bar{x}_r^-} \frac{\upsilon_r}{K_r} = f_r(\bar{x}_r)\,\frac{\lambda_r(\bar{x}_r)}{f_r(\bar{x}_r)}\,\frac{[\{\lambda_r(x_r)\}] + \lambda'_r(\bar{x}_r)\,\bar{x}_r}{-\lambda_r(\bar{x}_r)} = f_r(\bar{x}_r)\,\sigma_r(\bar{x}_r) \equiv c_r$ where $[\{\lambda_r(x_r)\}] \approx 0$ again. A unit tax υ_r is also proportional to ($\propto$) his multiplier K_r as $\upsilon_r \propto K_r$ through $\upsilon_r = c_r\,K_r$ with the above constant $c_r \equiv f_r(\bar{x}_r)\,\sigma_r(\bar{x}_r)$ in the neighborhood of $x_r \leq \bar{x}_r$ for an infinitesimal unit tax of $\upsilon_r \geq 0$ for all $r = 1, 2, \cdots, n$. Then, in this case, $\upsilon_r \not\propto \sigma_r(\bar{x}_r) = \frac{1}{\rho_r(\bar{x}_r)} + \frac{1}{\varepsilon_r(\bar{x}_r)}$, either.

all elements i, j, and $r \in \{s \mid s = 1, 2, \cdots, n\}$. To see it, *e.g.*, recall equation (29) for the l'Hôpital's rule, which is tractable for us to convey it into

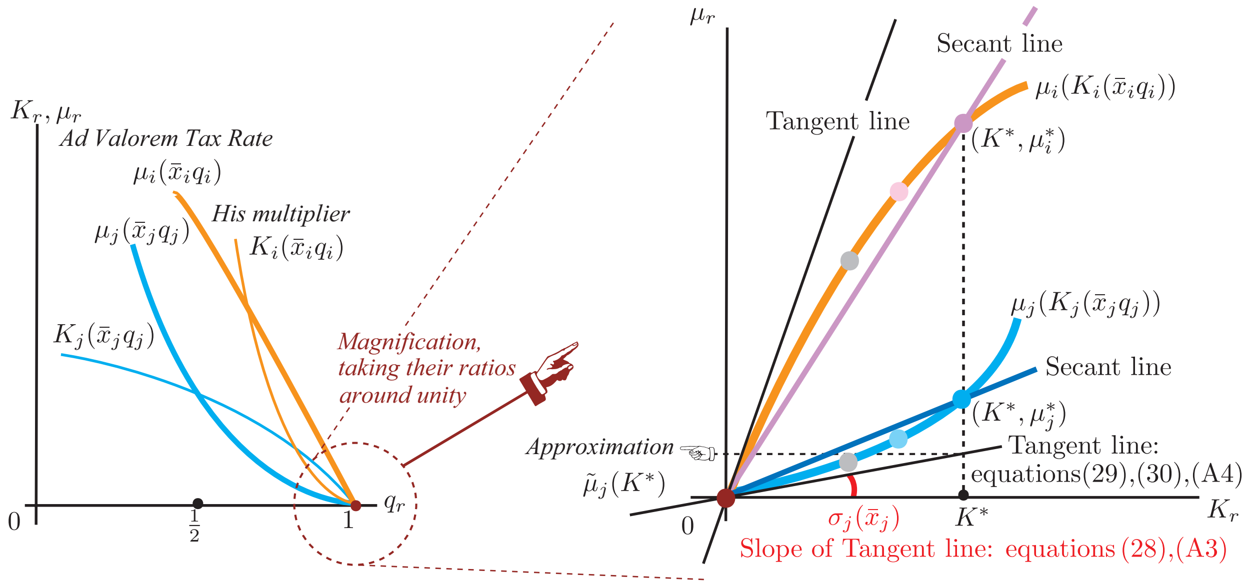

Figure 2: Relationships among Secant Lines and Tangent Lines

$$(32) \quad \frac{\mu_r'(\bar{x}_r)}{K_r'(\bar{x}_r)} = \frac{d\,\mu_r/d\,x_r}{d\,K_r/d\,x_r}\Big|_{x_r=\bar{x}_r} = \frac{d\,\mu_r}{d\,K_r}\Big|_{x_r=\bar{x}_r} = \frac{d\,\mu_r}{d\,K_r}\Big|_{\mu_r=K_r=0} = c_r.$$

The straight line $\mu_r = \sigma_r(\bar{x}_r)\,K_r$ in equations (29), (30), and (A4) is nothing but a tangent line at the origin (0, 0) of $K_r = \mu_r = 0$ in the K_r-μ_r plane, then its total derivative is calculated as $d\,\mu_r = \sigma_r(\bar{x}_r)\,d\,K_r$. Let $\mu_r^* = \mu_r(x_r^*)$ and $K^* = K_r(x_r^*)$ be an optimal ad valorem tax rate and its corresponding his multiplier, respectively. It is quite well known in calculus that the derivative is the limit of a difference quotient:[16] That is, a secant line, which is also a straight line that passes through two points like (0, 0) and (K^*, μ_r^*) in figure 2, tends close to the tangent line as the secant point (K^*, μ_r^*) tends close to (0, 0), or in the Gottfried Wilhelm von Leibniz's notation that when a difference $\Delta K^* \equiv K^* - 0 = K^*$ became the infinitesimal $d\,K$ or $d\,K_r$, a difference $\Delta\,\mu_r^* \equiv \mu_r^* - 0 = \mu_r^*$ simultaneously became the infinitesimal $d\,\mu_r$ and their difference quotient of $\dfrac{\Delta\,\mu_r^*}{\Delta\,K^*} = \dfrac{\mu_r^* - 0}{K^* - 0} = \dfrac{\mu_r^*}{K^*}$ became $\dfrac{d\,\mu_r}{d\,K}$, which were not an approximation to the derivative any more but exactly the same as the derivative itself. In addition to this Leibniz's notation, remind any ratio $\dfrac{\mu_r^*}{K^*}$ of the secant line belongs to such an average function around (0, 0) that every ratio of ratios like $\dfrac{\mu_i^*}{\mu_j^*} = \dfrac{\mu_i^*}{K^*}\dfrac{K^*}{\mu_j^*}$ cannot even estimate a precise value from it, and so that one had better use our equation (4) to do so if she or he wants a precise one. Thus, any ratio of optimal indirect taxes such as $\dfrac{\mu_i^*}{\mu_j^*}$ and $\dfrac{\upsilon_i^*}{\upsilon_j^*}$ with an "infinitesimal" K^*, say $d\,K^*$ given without solving the multiplier $\kappa = (-K)$ no longer implicates in the optimal ratio of them but just an approximation to its derivative like $\dfrac{d\,\mu_i^*}{d\,\mu_j^*} = \dfrac{d\,\mu_i^*}{d\,K^*}\,\dfrac{d\,K^*}{d\,\mu_j^*}$ at most.

[16]See, *e.g.*, Jerrold Marsden and Alan Weinstein (1985, pp.49-75) for the limit of a difference quotient, a secant line, and the Leibniz's notation.

Finally, what we have shown here is the Ramsey tax rules unfortunately told us for eight decades that a variable is proportional to ($\propto$) a coefficient: *I. e.*, they should not have misled us into believing as if $y \propto 2$ in regard to a tangent line of $y = 2\,x$ with two variables x, y and a constant coefficient $c_j = 2$; which may represent the j-th commodity market in figure 2 with a demand function $\phi_j = \dfrac{\gamma_j}{x_j}$, $\gamma_j > 0$, a supply function $f_j = a_j\, x_j$, $a_j > 0$, these price elasticities $\rho_j = \varepsilon_j = 1$, and so $c_j = \sigma_j \equiv \dfrac{1}{\rho_j} + \dfrac{1}{\varepsilon_j} = 2$ from equation (28).

II. 'Equal Sacrifice' Rules and Closed-form Solutions

A. *A Treatment of Our Lagrange-multiplier* κ

Let us rewrite the first-order condition from the Ramsey's indirect taxation, or equations (15) and (16):

$$(15)\quad \frac{\lambda_r}{\lambda_r + \lambda_r'\, x_r} = \kappa \quad \text{for } r = 1, 2, \cdots, n;$$

$$(16)\quad R = \sum_{r=1}^{n} R_r = \sum_{r=1}^{n} \lambda_r\, x_r,$$

in which κ is the multiplier and R_r is a government's tax revenue raised from the r-th market. Because we are able to interpret her tax revenue R_r as a 'sacrifice' of an indirect tax payment from that market, denote here again its market-surplus ms_r in equation (14) by

$$(33)\quad ms_r \equiv \int_0^{x_r} \lambda_r(s_r)\, ds_r,$$

then equation (15) provides us with an 'equal-marginal-sacrifice' rule of

PROPOSITION 2: *In a post-tax equilibrium, the r-th market's marginal market-surplus ms_r with respect to its tax-payment R_r to the government, or*

$$(34)\quad \frac{d\,ms_r}{d\,R_r} = \frac{d\,ms_r}{d\,x_r}\,\frac{1}{d\,R_r/d\,x_r} = \kappa \le 0$$

equals each other to κ for all $r = 1, 2, \cdots, n$ owing to both of numerator and denominator in equation (15), or to that $\dfrac{d\,ms_r}{d\,x_r} = \lambda_r \ge 0$ and $\dfrac{d\,R_r}{d\,x_r} = \lambda_r + \lambda_r'\,x_r < 0$ from (33), (1), and (13), respectively.

Proof. See appendix A in details. □

So, we have the 'equal-marginal-sacrifice' rule for equation (14) that the r-th market's 'sacrifice' or negative market-surplus ($-ms_r$) equally goes up by $-\kappa$ units when the tax-payment R_r goes up by one unit. Besides, the length λ_r in equation (1) is so easily translated into an average function of the tax-payment R_r as

$$(35)\quad \lambda_r = \frac{\lambda_r\, x_r}{x_r} = \frac{R_r}{x_r} \quad \text{for } r = 1, 2, \cdots, n$$

with equation (6) that equation (15) can give us another one, say an 'equal-elasticity-sacrifice' rule of

COROLLARY 2: *At the very same time, an elasticity of post-tax equilibrium quantity x_r of the r-th market with respect to its tax-payment R_r to the government, or the 'sacrifice' elasticity of post-tax quantity as*

$$(36)\quad \frac{d\,ms_r}{d\,x_r}\,\frac{1}{d\,R_r/d\,x_r} = \frac{d\,x_r/x_r}{d\,R_r/R_r} = \kappa \le 0$$

equals each other to κ *for all* $r = 1, 2, \cdots, n$ *due to that* $\dfrac{d\,ms_r}{d\,x_r} = \lambda_r = \dfrac{R_r}{x_r}$ *from equations (33) and (35).*

Proof. See appendix A for details. □

Now, we have the 'equal-elasticity-sacrifice' rule for equation (14) that the r-th market's 'sacrifice' or negative post-tax quantity $(-x_r)$ equally goes up by $-\kappa\%$ when the tax-payment R_r goes up by 1 %. Alternatively, it is easy to observe from equations (36) and (A5) of the unit-less quantity $q_r = \dfrac{x_r}{\bar{x}_r}$ that the above 'equal-elasticity-sacrifice' rule has another expression of

$$(37)\quad \kappa = \frac{d\,x_r/x_r}{d\,R_r/R_r} = \frac{d\,q_r/q_r}{d\,R_r/R_r} = \frac{R_r/q_r}{d\,R_r/d\,q_r} \qquad \text{for } r = 1, 2, \cdots, n$$

because $x_r = \bar{x}_r\,q_r$ as well as its sufficiently small increment $d\,x_r = \bar{x}_r\,d\,q_r$ are calculated from equation (A5).

To completely solve the first-order condition of equations (15) and (16), we need a few more steps than in linear cases since equation (16) is a non-linear function of x_r: As its first step, we show equation (15) in terms of a common κ only, or we ought to have n functions x_r of κ by computing some inverse functions as

$$(38)\quad x_r = x_r(\kappa) \qquad \text{for } r = 1, 2, \cdots, n;$$

Next, by plugging up each equation (38) in both λ_r and x_r of equation (16), we obtain a function of κ or

$$(39)\quad R = \sum_{r=1}^{n} \lambda_r(x_r(\kappa))\,x_r(\kappa) = \sum_{r=1}^{n} R_r(\kappa).$$

At last, equation (39) provides us with at least one root to be examined against a second-order condition of a bordered Hessian that we put in our appendix B. In the case of $q_r = \dfrac{x_r}{\bar{x}_r}$ from equation (A5), we also have such n functions q_r of κ as $q_r = q_r(\kappa)$ for $r = 1, 2, \cdots, n$ that equation (39) can be expressed as

$$(40)\quad R = \sum_{r=1}^{n} \lambda_r(\bar{x}_r\,q_r(\kappa))\,\bar{x}_r\,q_r(\kappa) = \sum_{r=1}^{n} R_r(\kappa).$$

If we can successfully choose appropriate inverse functions for equation (38) and obtain equation (39) or (40) before examining some roots as optimal candidates against a second-order condition, then the following proposition is available for an optimal solution of the Ramsey's indirect taxation in equation (14).

PROPOSITION 3: *Let us rewrite equations (39) and (40) as a tax revenue function* t *of the multiplier* κ*:*

$$(41)\quad t = t(\kappa) = \sum_{r=1}^{n} R_r(\kappa);\ \textit{monotone decreasing as } \frac{d\,t}{d\,\kappa} < 0 \textit{ for } \kappa \leq 0,$$

then there exists a unique optimal solution κ^* *at* $\kappa = \kappa^* \leq 0$*, corresponding to the government's constraint* R*;* $0 \leq R < \bar{R} \equiv \sum_{r=1}^{n} R_r^{max}$*, in which* $\bar{R}$ *is an 'ability' of all markets, or a sum of every market's 'ability'* R_r^{max} *of its maximum amount potentially payable to the government by the* r*-th commodity market in footnote 9.*

Proof. See appendix A for this proof. □

Consider a parametric example with n-commodity markets, in each of which a government faces not only an inverse demand function

(42) $\phi_r = \phi_r(x_r) \equiv -\alpha_r\, x_r + \beta_r \quad$ for $r = 1, 2, \cdots, n$

but also a supply function

(43) $f_r = f_r(x_r) \equiv a_r\, x_r + b_r \quad$ for $r = 1, 2, \cdots, n$

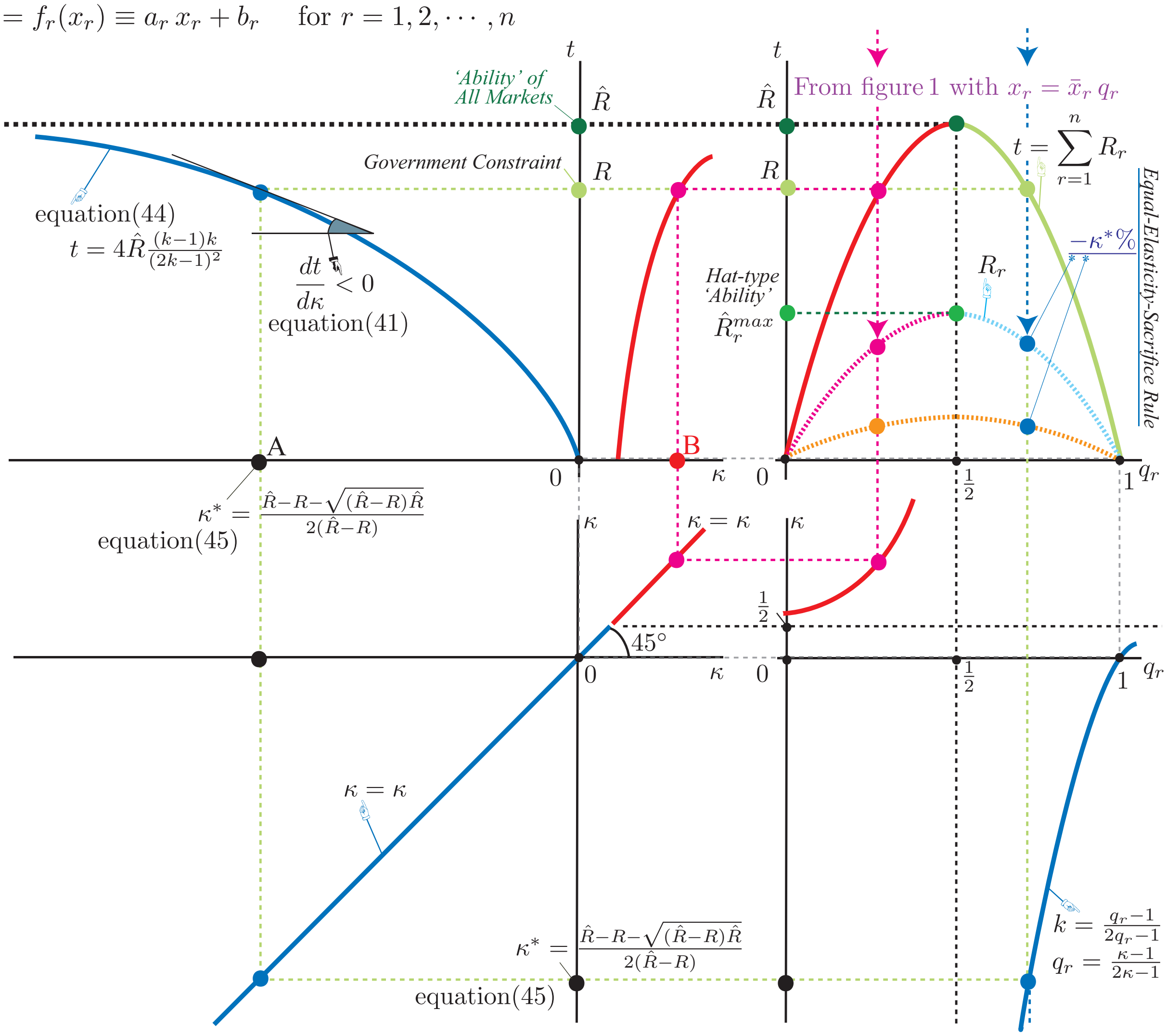

Figure 3: Corollary 3; optimal κ^* for a Certain R Attains at Point • A, not at • B

with parameters $\alpha_r > 0$, $\beta_r > b_r \geq 0$, and $a_r \geq 0$ but $a_r = b_r \neq 0$ at the same time, and she imposes an ad valorem tax rate μ_r on the r-th commodity in order to collect her tax revenue by an amount of R out of the 'ability' of all markets $\bar{R}$ rewritten here by $\bar{R} = \hat{R} \equiv \sum_{r=1}^{n} \hat{R}_r^{max} = \sum_{r=1}^{n} \dfrac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$ as seen in equation (8).

COROLLARY 3: *As its tax revenue function t of the multiplier κ becomes*

(44) $t = t(\kappa) = 4\,\hat{R}\,\dfrac{\kappa\,(\kappa-1)}{(2\,\kappa-1)^2}$ *and so that* $\dfrac{d\,t}{d\,\kappa} = \dfrac{4\,\hat{R}}{(2\,\kappa-1)^3} < 0$ *for* $\kappa \leq 0$,

there exists a unique optimal solution κ^* as demonstrated in figure 3 of

$$(45)\quad \kappa^* = \frac{\hat{R} - R - \sqrt{(\hat{R} - R)\,\hat{R}}}{2\,(\hat{R} - R)} \leq 0$$

for a government tax revenue R to be gathered out of the 'ability' of all markets $\hat{R}$. Then, an optimal ad valorem tax rate μ_r^* on the r-th commodity is computed as

$$(46)\quad \mu_r^* = \frac{(\alpha_r + a_r)\,(\beta_r - b_r)\,(\sqrt{\hat{R}} - \sqrt{\hat{R} - R})}{2\,(\alpha_r + a_r)\,b_r\,\sqrt{\hat{R}} + a_r\,(\beta_r - b_r)\,(\sqrt{\hat{R}} + \sqrt{\hat{R} - R})}$$

for the government to obtain her optimal tax revenue R_r^* of

$$(47)\quad R_r^* = \frac{\hat{R}_r^{max}}{\hat{R}} R$$

where $\hat{R} \equiv \sum_{r=1}^{n} \hat{R}_r^{max}$ and $\hat{R}_r^{max} \equiv \dfrac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$, which is Hat-type 'ability' in the r-th market with $0 \leq R < \hat{R} \equiv \dfrac{1}{4} \sum_{r=1}^{n} \dfrac{(\beta_r - b_r)^2}{\alpha_r + a_r}$, the 'ability'of all markets $\bar{R}$ in this case.

Proof. Even though the above figure 3 that reflects what is going on is likely to give us a pictorial proof, see appendixes A and B✍❶❷❸ in details. □

It is easy to observe in equation (45) as a parametric closed-form solution that we have not only the 'equal-marginal-sacrifice' rule in equation (34) but the 'equal-elasticity-sacrifice' rule in equation (36), respectively. It is also easy to see in equation (46) directly derived from equation (4) that every optimal ad valorem tax rate μ_r^* seems so complicated that no one is now able to insist on the Ramsey's 'reciprocal elasticity rule' and his 'ratio rule' any more as stated in proposition 1 and its corollary 1. Moreover, it is interesting for us to contemplate in equation (47) that every market should pay its optimal indirect tax R_r^* for a certain amount of R at a weighted 'ability' portion $\dfrac{\hat{R}_r^{max}}{\hat{R}}$ or a ratio of the Hat-type 'ability' $\hat{R}_r^{max}$ of the r-th market to that of all markets $\hat{R}$, say a weighted 'ability' rule, and so that the government can obtain her revenue R from among n markets in descending order of the 'ability' $\hat{R}_r^{max}$ of the biggest amount potentially payable to her as the r-th market's indirect tax.[17] So, it can be seen in n affine markets as put in Ramsey (1927, pp.56-8) that the weighted 'ability' rule seems to prevail over his 'reciprocal elasticity rule' as well as its 'ratio rule' so that it would be better compare with the r-th market's 'ability' $\hat{R}_r^{max}$ each other for all $r = 1, 2, \cdots, n$ rather than price elasticities of demand ρ_r and supply ε_r.

C. *An Example with Hyperbolic Demand Functions*

Consider another example with n-commodity markets where a government meets a non-linear inverse demand function of

[17] Instead of equation (46), *ceteris paribus*, an optimal unit tax υ_r^* for the r-th commodity in those n homogeneous affine markets is computed as $\upsilon_r^* = \frac{\beta_r - b_r}{2} \frac{\hat{R} - \sqrt{(\hat{R} - R)\,\hat{R}}}{\hat{R}}$ from footnote 8 or a numerator of equation (A18) with equation (A17) in order to collect an amount of R_r^* in equation (47) as her tax revenue for the r-th commodity market.

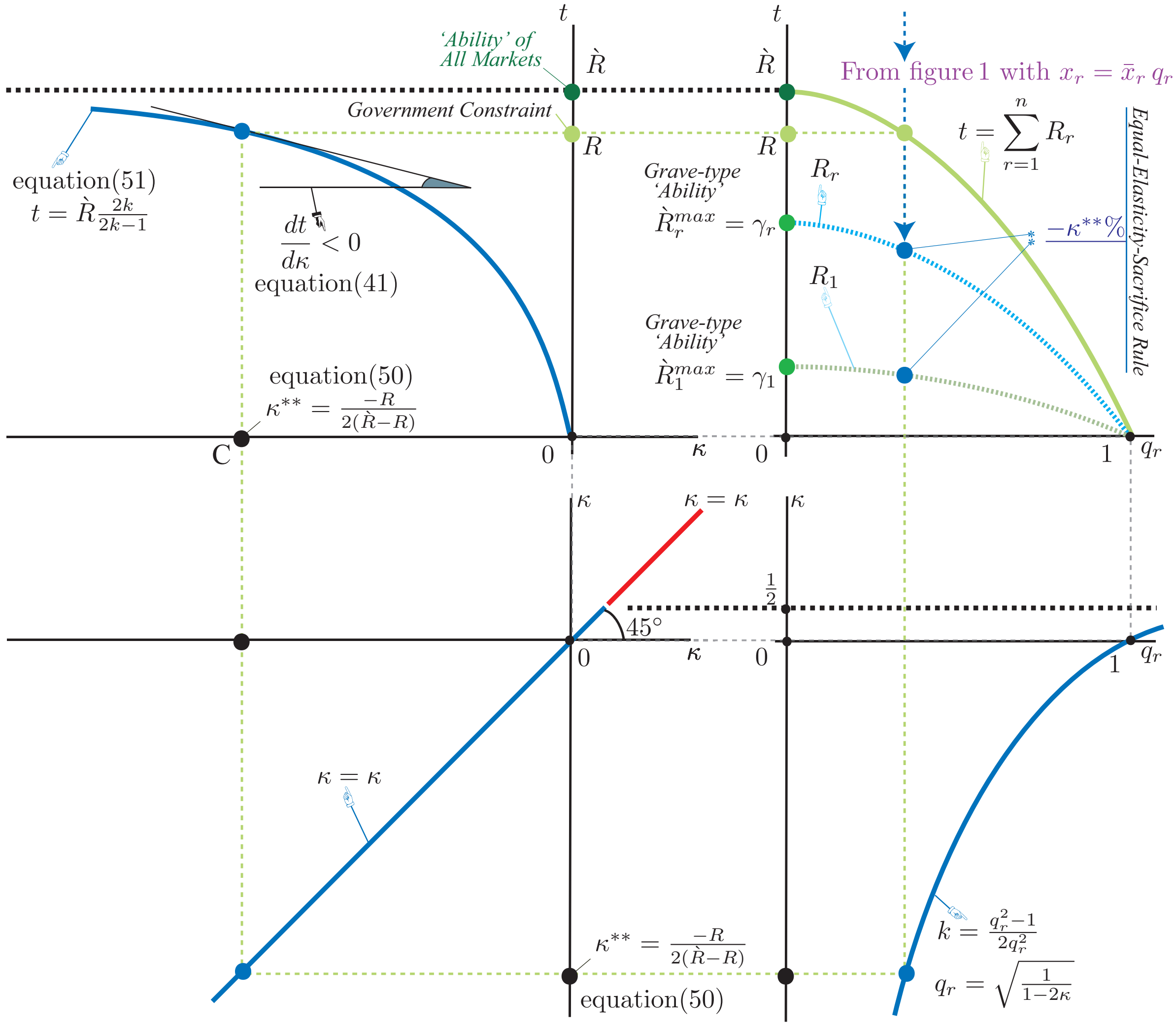

Figure 4: Corollary 4; optimal κ^{**} for a Given R Holds at Point • C

(48) $\phi_r = \phi_r(x_r) \equiv \dfrac{\gamma_r}{x_r}, \gamma_r > 0 \quad$ for $r = 1, 2, \cdots, n$

whose price elasticity ρ_r of demand takes a value of unity everywhere in its domain of equation (2), say $\rho_r^{**} \equiv -\dfrac{\phi_r(x_r)}{\phi_r'(x_r)\, x_r} = 1$. Meanwhile, she has a linear supply function in each market as

(49) $f_r = f_r(x_r) \equiv a_r\, x_r, a_r > 0 \quad$ for $r = 1, 2, \cdots, n$

whose price elasticity ε_r of supply also takes a value of unity anywhere in its domain of equation (2), or $\varepsilon_r^{**} \equiv \dfrac{f_r(x_r)}{f_r'(x_r)\, x_r} = 1$. As discussed in section II.D, the sum function σ_r in equation (28) of reciprocals of those price elasticities takes a value of two as $\sigma_r^{**} \equiv \dfrac{1}{\rho_r^{**}} + \dfrac{1}{\varepsilon_r^{**}} = 2$ for all $r = 1, 2, \cdots$, and n. In this case, the government levies an ad valorem tax rate μ_r on the r-th commodity in order to collect her tax revenue by an amount of R out of the 'ability' of all markets $\bar{R} = \grave{R} \equiv \sum_{r=1}^{n} \grave{R}_r^{max} = \sum_{r=1}^{n} \gamma_r$ as shown in equation (10).

COROLLARY 4: *There exists a unique optimal solution* κ^{**} *of*

(50) $\kappa^{**} = \dfrac{-R}{2\,(\grave{R} - R)} \leq 0$ *for her tax revenue* $R; 0 \leq R < \grave{R} \equiv \sum_{r=1}^{n} \gamma_r$

because its tax revenue function t of the multiplier κ becomes

$$(51)\quad t = t(\kappa) = 2\,\grave{R}\,\frac{\kappa}{2\,\kappa - 1} \text{ and so that } \frac{d\,t}{d\,\kappa} = \frac{-2\,\grave{R}}{(2\,\kappa - 1)^2} < 0 \text{ for } \kappa \leq 0.$$

Hence, an optimal ad valorem tax rate μ_r^{**} on the r-th commodity is computed as

$$(52)\quad \mu_r^{**} = \frac{R}{\grave{R} - R},\ 0 \leq R < \grave{R} \equiv \sum_{r=1}^{n} \grave{R}_r^{max} \equiv \sum_{r=1}^{n} \gamma_r$$

to have an optimal tax revenue R_r^{**} of

$$(53)\quad R_r^{**} = \frac{\grave{R}_r^{max}}{\grave{R}} R$$

where $\grave{R} \equiv \sum_{r=1}^{n} \grave{R}_r^{max}$ and $\grave{R}_r^{max} \equiv \gamma_r$ given in the r-th Grave-type market for all $r = 1, 2, \cdots, n$.[18]

Proof. See appendixes A and B✍❹❺❻ in details with the above figure 4. □

Other things being equal, suppose only equation (49) were replaced by the following horizontal supply function of

$$(54)\quad f_r(x_r) \equiv b_r > 0 \quad \text{for } r = 1, 2, \cdots, n$$

in corollary 4, then its price elasticity ε_r of supply diverges to infinity (∞) in its domain of equation (2), or $\bar{\varepsilon}_r^{**} \equiv \dfrac{f_r(x_r)}{f_r'(x_r)\,x_r} \to \infty$ letting its reciprocal $\dfrac{1}{\bar{\varepsilon}_r^{**}}$ converged to zero, and so making the sum function σ_r in equation (28) taken a value of unity as $\bar{\sigma}_r^{**} \equiv \dfrac{1}{\rho_r^{**}} + \dfrac{1}{\bar{\varepsilon}_r^{**}} = 1$ for all $r = 1, 2, \cdots, n$.

COROLLARY 5: *The government has a monotone decreasing tax function of $t = t(\kappa) = \grave{R}\,\dfrac{\kappa}{\kappa - 1}$, which is different from equation (51) as well as an optimal solution $\bar{\kappa}^{**}$ of*

$$(55)\quad \bar{\kappa}^{**} \equiv \frac{-R}{\grave{R} - R} \leq 0,$$

which is also a little bit different from equation (50); however, she has exactly the same optimal ad valorem tax rate $\bar{\mu}_r^{**}$ as in equation (52):[19]

$$(56)\quad \bar{\mu}_r^{**} = \frac{R}{\grave{R} - R},\ 0 \leq R < \grave{R} \equiv \sum_{r=1}^{n} \grave{R}_r^{max} \equiv \sum_{r=1}^{n} \gamma_r$$

on the r-th commodity with its market's tax revenue $\bar{R}_r^{**} \equiv \dfrac{\grave{R}_r^{max}}{\grave{R}} R$ as equation (53).

Proof. See appendixes A and B✍❼❽❾ for this proof. □

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[18] For this example, an optimal unit tax v_r^{**} is given as $v_r^{**} = \sqrt{\gamma_r\,a_r}\,R \,/ \sqrt{(\grave{R} - R)\,\grave{R}}$ according to footnote 8 or a numerator of equation (A29) with equation (A27).

[19] With horizontal supply functions of equation (54), from footnote 8 or a numerator of equation (A30) with equation (A28), an optimal unit tax $\bar{v}_r^{**}$ turns to be $\bar{v}_r^{**} = b_r\,R \,/\, \grave{R}$.

In equations (50) and (55) of parametric solutions, we can see both the 'equal-marginal-sacrifice' rule in equation (34) and the 'equal-elasticity-sacrifice' rule in equation (36). It is interesting to see in equations (52) and (56) that the government imposes exactly the identical optimal rate μ_r^{**} or $\bar{\mu}_r^{**}$ on the all markets,[20] and so with equation (28) that no matter how the sum function σ_r of reciprocals of price elasticities might take a value of $\sigma_r^{**} = 2$ or $\bar{\sigma}_r^{**} = 1$, a ratio of tax rates from among μ_r^{**} or $\bar{\mu}_r^{**}$ always becomes unity as

$$(57)\quad \frac{\mu_i^{**}}{\mu_j^{**}} = \frac{\bar{\mu}_i^{**}}{\bar{\mu}_j^{**}} = \frac{R}{\grave{R} - R}\,\frac{\grave{R} - R}{R} = 1$$

for all elements i, j, and $r \in \{s \mid s = 1, 2, \cdots, n\}$. So, equation (57) immediately tells us that the 'ratio rule' discussed in corollary 1 malfunctions again. On the other hand, it is also interesting to see in equation (53) that the r-th commodity market should pay its optimal indirect tax R_r^{**} for an amount of R at a weighted 'ability' portion $\dfrac{\grave{R}_r^{max}}{\grave{R}}$ or a ratio of the Grave-type 'ability' $\grave{R}_r^{max}$ of the r-th market to that of all markets $\grave{R}$, say a weighted 'ability' rule, and so that the government can obtain her revenue R from among n markets in descending order of the Grave-type 'ability' $\grave{R}_r^{max}$, which is the biggest amount potentially payable to her as the r-th market's indirect tax. Since the weighted 'ability' rule prevails over both of the 'reciprocal elasticity rule' and the 'ratio rule' in n markets again, we had better compare with the r-th market's 'ability' $\grave{R}_r^{max}$ each other for all $r = 1, 2, \cdots, n$ instead of price elasticities of demand $\rho_r = 1$ and supply ε_r.

D. *An Example with Mixed Markets*

Previously on a few of examples, n markets are kinds of homogeneous where functions for demand or supply are supposed to take identical forms but these parameters' values are supposed to be different from each other. So, consider here heterogeneous three-commodity markets where a government faces three inverse demand functions as

$$(42)\quad \phi_r = \phi_r(x_r) \equiv -\alpha_r\, x_r + \beta_r \qquad \text{for } r = 1$$

$$(48)\quad \phi_r = \phi_r(x_r) \equiv \frac{\gamma_r}{x_r} \qquad \text{for } r = 2, 3$$

as well as the following three supply functions of

$$(43)\quad f_r = f_r(x_r) \equiv a_r\, x_r + b_r \qquad \text{for } r = 1$$

$$(49)\quad f_r = f_r(x_r) \equiv a_r\, x_r \qquad \text{for } r = 2, 3$$

with parameters $\alpha_r > 0$, $\beta_r > b_r \geq 0$, $\gamma_r > 0$, and $a_r \geq 0$ but $a_r = b_r \neq 0$ simultaneously, and she imposes an ad valorem tax rate μ_r on the r-th commodity in order to collect her tax revenue by an amount of R out of the 'ability' of all markets $\bar{R}$ denoted here by $\hat{R}_{mix}^{max} \equiv \hat{R}_1^{max} + \grave{R}_2^{max} + \grave{R}_3^{max}$ where a Hat-type $\hat{R}_1^{max} \equiv \dfrac{(\beta_1 - b_1)^2}{4\,(\alpha_1 + a_1)}$ and a Grave-type $\grave{R}_r^{max} \equiv \gamma_r$ for $r = 2$ and 3, as shown in equations (8) and (10), respectively.

[20] *N. b.*, these optimal rates μ_r^{**} and $\bar{\mu}_r^{**}$ do not seem to be influenced by supply functions such as equations (49) and (54) because every market's 'ability' of $\grave{R}_r^{max} = \gamma_r$ here in equation (53), which is the maximum amount potentially payable to the government as the r-th market's indirect tax, consists of a parameter γ_r only in a demand equation (48).

COROLLARY 6: *As its tax revenue function t of the multiplier κ becomes*

$$(58)\quad t = t(\kappa) = 4\,\hat{R}_1^{max}\,\frac{\kappa\,(\kappa-1)}{(2\,\kappa-1)^2} + 2\,\gamma_2\,\frac{\kappa}{2\,\kappa-1} + 2\,\gamma_3\,\frac{\kappa}{2\,\kappa-1}$$

in this example, then there exists a unique optimal solution $\kappa^*_{mix} \le 0$ of

$$(59)\quad \kappa^*_{mix} = \frac{2\,(\hat{R}^{max}_{mix} - R) - (\gamma_2+\gamma_3) - \sqrt{4\,(\hat{R}^{max}_{mix} - R)\,\hat{R}^{max}_1 + (\gamma_2+\gamma_3)^2}}{4\,(\hat{R}^{max}_{mix} - R)}$$

for her tax revenue R; $0 \le R < \hat{R}^{max}_{mix} = \dfrac{1}{4}\,\dfrac{(\beta_1 - b_1)^2}{\alpha_1 + a_1} + \gamma_2 + \gamma_3$.

Proof. See appendix A for this proof. □

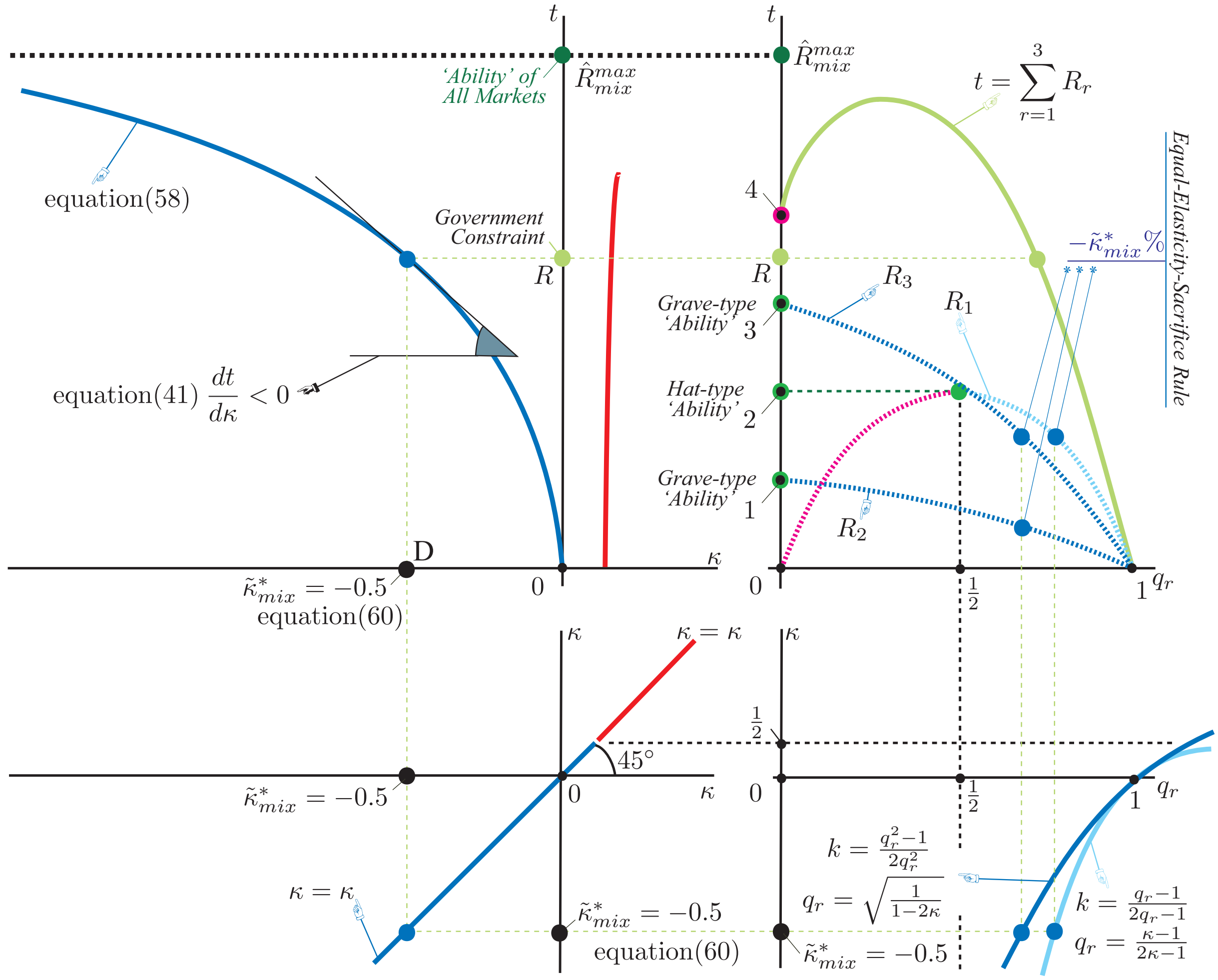

Figure 5: Corollary 6; optimal $\tilde{\kappa}^*_{mix}$ for a Certain R Attains at Point • D

It is interesting to see in a quadratic equation (A31) of appendix A as

$$(\text{A31})\quad 4\,(\hat{R}^{max}_{mix} - R)\,\kappa^2 - 2\,(\hat{R}^{max}_{mix} + \hat{R}^{max}_1 - 2\,R)\,\kappa - R = 0,$$

which is derived from equation (58) by being equal to R that the more heterogeneities of commodity markets increase, the higher order in a polynomial equation of κ shows up: For example, if we replace the last term $2\,\gamma_3\,\dfrac{\kappa}{2\,\kappa-1}$ in equation (58) by $\gamma_3\,\dfrac{\kappa}{\kappa-1}$ in corollary 5 or equation (A26), we must solve a cubic equation of

$$4\,\hat{R}^{max}_1\,\frac{\kappa\,(\kappa-1)}{(2\,\kappa-1)^2} + 2\,\gamma_2\,\frac{\kappa}{2\,\kappa-1} + \gamma_3\,\frac{\kappa}{\kappa-1} = R;$$

perhaps without any parametrically trim root nor an elegant algebraic solution by the factor theorem: Or, the higher order a polynomial equation has, algebraically speaking, the less opportunity to use the general formula of roots we obtain.

Let us make up a numerical example to ease in us for calculations as seen in figure 5, namely, not merely to make each market's 'ability' taken a value of $\hat{R}_1^{max} = 2$, $\grave{R}_2^{max} = 1$, and $\grave{R}_3^{max} = 3$ with parameters $a_3 = \frac{1}{3}$, $\alpha_1 = \gamma_2 = a_1 = b_1 = 1$, $\gamma_3 = 3$, $a_2 = 4$, and $\beta_1 = 5$, but also to make the government collect R as her tax revenue by an amount of $R = 3.5$ out of the 'ability' of all markets $\hat{R}_{mix}^{max} = \frac{1}{4} \frac{(\beta_1 - b_1)^2}{\alpha_1 + a_1} + \gamma_2 + \gamma_3 = 6$. Then, every market's initial equilibrium quantity $\bar{x}_r$ before tax, an upper limit of the domain in equation (2) for $r = 1$, 2, and 3, is calculated as $\bar{x}_1 = 2$, $\bar{x}_2 = 0.5$, and $\bar{x}_3 = 3$, respectively, by equations (42), (43), (48), and (49). From equation (59), the government can numerically determine an optimal value of the multiplier denoted by $\tilde{\kappa}_{mix}^*$ as

$$(60) \quad \tilde{\kappa}_{mix}^* \equiv \frac{2\,(6-3.5) - (1+3) - \sqrt{4\,(6-3.5)\,2 + (1+3)^2}}{4\,(6-3.5)} = \frac{1-6}{10}.$$

Substitute this equation (60) or $\tilde{\kappa}_{mix}^* = -0.5$ into equations (A13) and (A23) to obtain unit-less quantities q_r's in equation (A5) as $q_1 = \frac{3}{4}$ for market 1 and as $q_2 = q_3 = \frac{\sqrt{2}}{2}$ for markets 2 and 3, then we have equilibrium quantities after tax as $\tilde{x}_1^* = \bar{x}_1\, q_1 = 1.5$, $\tilde{x}_2^* = \bar{x}_2\, q_2 = \frac{\sqrt{2}}{4}$, and $\tilde{x}_3^* = \bar{x}_3\, q_3 = \frac{3\sqrt{2}}{2}$.

Next, equations (A18) and (A29) provide us with optimal ad valorem tax rates $\tilde{\mu}_1^* = 0.4$ as 40% and $\tilde{\mu}_2^* = \tilde{\mu}_3^* = 1$ as 100%, respectively; whereas according to footnote 8, numerators of equations (A18) and (A29) provide us with not only optimal unit taxes but also vertical lengths as $\tilde{v}_1^* = \tilde{\lambda}_1^* = 1$, $\tilde{v}_2^* = \tilde{\lambda}_2^* = \sqrt{2}$, and $\tilde{v}_3^* = \tilde{\lambda}_3^* = \frac{\sqrt{2}}{2}$.

Now, equation (6) is able to tell us that the government raises her tax revenue by $R = 3.5$ as $\tilde{R}_1^* = \tilde{\lambda}_1^*\, \tilde{x}_1^* = 1.5$, $\tilde{R}_2^* = \tilde{\lambda}_2^*\, \tilde{x}_2^* = 0.5$, and $\tilde{R}_3^* = \tilde{\lambda}_3^*\, \tilde{x}_3^* = 1.5$ whose results can be checked out by substituting $\tilde{\kappa}_{mix}^* = -0.5$ in equation (60) into each term of equation (58): *I.e.*, $\tilde{R}_1^* = 4\,\hat{R}_1^{max} \frac{\tilde{\kappa}_{mix}^*\,(\tilde{\kappa}_{mix}^* - 1)}{(2\,\tilde{\kappa}_{mix}^* - 1)^2}$, $\tilde{R}_2^* = 2\,\gamma_2 \frac{\tilde{\kappa}_{mix}^*}{2\,\tilde{\kappa}_{mix}^* - 1}$, and $\tilde{R}_3^* = 2\,\gamma_3 \frac{\tilde{\kappa}_{mix}^*}{2\,\tilde{\kappa}_{mix}^* - 1}$.

Finally, the following bordered principal minors:

$$|\bar{H}_2| \equiv \begin{vmatrix} 0 & -2 & -2\sqrt{2} \\ -2 & -4 & 0 \\ -2\sqrt{2} & 0 & -16 \end{vmatrix} ; \; |\bar{H}_3| \equiv \begin{vmatrix} 0 & -2 & -2\sqrt{2} & -\sqrt{2} \\ -2 & -4 & 0 & 0 \\ -2\sqrt{2} & 0 & -16 & 0 \\ -\sqrt{2} & 0 & 0 & -4/3 \end{vmatrix}$$

are satisfied with the second-order condition due to our appendix B ✌①②③.

Accordingly, we can see in both of equation (59) with a parametric closed-form solution and equation (60) with a numerical solution that this example is also satisfied with not only the 'equal-marginal-sacrifice' rule in equation (34) but also the 'equal-elasticity-sacrifice' rule in equation (36). However, it is interesting to see here that a weighted 'ability' rule does not always work although it really works for the previous examples.

III. Concluding Remarks

In this paper, we have had two purposes: One is a reexamination of Frank P. Ramsey's tax rules such as the 'reciprocal elasticity rule,' its 'ratio rule,' and the 'equal-marginal-sacrifice rule;' the other is seeking some parametric closed-form solutions. One of the reason why we have done so is that Ramsey (1927, p.56) showed us $\underline{\text{equation (11)}}$ $\mu_r = \dfrac{\theta(1/\varepsilon_r + 1/\rho_r)}{1-\theta/\rho_r}$, which should have been evaluated like our equation (C11) of $\mu_r = \dfrac{\theta(1/\varepsilon_r + 1/\rho_r)}{1-\theta/\rho_r} \to 0$ as $\theta \to 0^+$ where we underline to make it different from ours, but he did not show us a mathematical proof on his 'reciprocal elasticity rule' derived from it. Instead, he just verbally stated in his $\underline{\text{equation (12)}}$ as "infinitesimal $\cdots$ tax *ad valorem* on each commodity should be proportional to the sum of the reciprocals of its supply and demand elasticities." Since then, he and his followers have used what they should have proved and so made their rules from such an "infinitesimal" tax, believing without a closed-form solution that these rules could be perfectly valid for "a tax of 500% on whisky" by way of example in Ramsey (*ibid.*, p.60), and there that "the more complicated results $\cdots$ may well be valid under still wider conditions."

To achieve our purposes, we have used exactly the same model as Ramsey (1927, pp.55-8) who took the total-differential approach to having his first-order condition as seen in footnote 12. As shown in proposition 1 and appendixes A and C, the l'Hôpital's rule of the limit of the $\dfrac{0}{0}$ form as well as *Infinitesimal Calculus* reveal that the 'reciprocal elasticity rule' is not true at all: *I. e.*, there does not exist anywhere such a proportionality between an "infinitesimal" indirect tax and a sum of reciprocals of price elasticities of supply ε_r and demand ρ_r because the sum function $\sigma_r = \dfrac{1}{\varepsilon_r} + \dfrac{1}{\rho_r}$ in equation (28) is always too large to have the same order in the $\dfrac{0}{0}$ form as "infinitesimal" candidates of six equations (21) through (25), and (C3) with its limit; then the limit of the $\dfrac{0/0}{0/0}$ form with the Leibniz's notation shows us in corollary 1 and appendix A that any 'ratio rule' of optimal taxes such as $\dfrac{\mu_i^*}{\mu_j^*}$ $(\propto \dfrac{\rho_j^*}{\rho_i^*})$ ad valorem no longer implicates in an accurate one but an approximation to its derivative at most. If its accurate one is needed, our equation (4) is go for it.

Since Ramsey does not give us a rule like the Gossen's Second Law nor with rules concerning 'sacrifice' to a direct tax-payment, we have shown the 'equal-marginal-sacrifice rule' of an optimal indirect taxation in proposition 2 as well as the 'equal-elasticity-sacrifice rule' of that in corollary 2, respectively. Besides, denote by R_r^{max} 'ability' of the r-th market, which is the maximum amount potentially payable to a government as an indirect tax of it, and by $\bar{R}$ that of all markets. As discussed in corollaries 3 through 5, we have a weighted 'ability' rule, at which rate of $\dfrac{R_r^{max}}{\bar{R}}$ the government can obtain her tax revenue R in descending order of the biggest one R_r^{max} from among n markets. So, as the weighted 'ability' rule is more useful than the 'reciprocal elasticity rule' and its 'ratio rule' over there, we had better compare with the r-th market's 'ability' R_r^{max} each other for all $r = 1, 2, \cdots, n$ rather than price elasticities of demand ρ_r and supply ε_r.

As obtained in corollaries 3 through 6, moreover, let R be a government tax revenue that the n-commodity markets should pay to her, and we have several parametric closed-form solutions for optimal ad valorem tax rates μ_r as well as unit taxes υ_r in affine markets*, hyperbolic demand markets**, and mixed markets$^{\tilde{}*}$ as

$$\text{(46)}\quad \mu_r^* = \frac{(\alpha_r + a_r)\,(\beta_r - b_r)\,(\sqrt{\hat{R}} - \sqrt{\hat{R} - R})}{2\,(\alpha_r + a_r)\,b_r\,\sqrt{\hat{R}} + a_r\,(\beta_r - b_r)\,(\sqrt{\hat{R}} + \sqrt{\hat{R} - R})} \text{ with the `ability' of } \bar{R} \equiv \hat{R} = \sum_{r=1}^{n} \frac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)},$$

$$\text{(f17)}\quad \upsilon_r^* = (\beta_r - b_r)\left\{\hat{R} - \sqrt{(\hat{R} - R)\,\hat{R}}\right\}/(2\,\hat{R}) \text{ with the `ability' of all markets } \bar{R} \equiv \hat{R} = \sum_{r=1}^{n} \frac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)},$$

$$\text{(52)}\quad \mu_r^{**} = R/(\grave{R} - R) \text{ with the `ability' of all markets } \bar{R} \equiv \grave{R} = \sum_{r=1}^{n} \gamma_r,$$

$$\text{(f18)}\quad \upsilon_r^{**} = \sqrt{\gamma_r\,a_r}\,R/\sqrt{(\grave{R} - R)\,\grave{R}} \text{ with the `ability' of all markets } \bar{R} \equiv \grave{R} = \sum_{r=1}^{n} \gamma_r,$$

$$\text{(56)}\quad \bar{\mu}_r^{**} = R/(\grave{R} - R) \text{ with the `ability' of all markets } \bar{R} \equiv \grave{R} = \sum_{r=1}^{n} \gamma_r,$$

$$\text{(f19)}\quad \bar{\upsilon}_r^{**} = b_r\,R/\grave{R} \text{ with the `ability' of all markets } \bar{R} \equiv \grave{R} = \sum_{r=1}^{n} \gamma_r,$$

$$\text{(cv̈)}\quad \tilde{\mu}_1^* = 40\%, \text{ and } \tilde{\mu}_2^* = \tilde{\mu}_3^* = 100\% \text{ with } R = 3.5 < \hat{R}_{mix}^{max} = 6 : \hat{R}_1^{max} = 2; \grave{R}_2^{max} = 1; \text{ and } \grave{R}_3^{max} = 3,$$

$$\text{(cv̈)}\quad \tilde{\upsilon}_1^* = 1, \tilde{\upsilon}_2^* = \sqrt{2}, \text{ and } \tilde{\upsilon}_3^* = \sqrt{2}/2 \text{ with } R = 3.5 < \hat{R}_{mix}^{max} = 6 : \hat{R}_1^{max} = 2; \grave{R}_2^{max} = 1; \text{ and } \grave{R}_3^{max} = 3$$

where 'f' in equation number # or (f#) stands for an initial 'f' of footnote and 'cv̈' in that for corollary 6, and parameters are given nearby equations (#), (f#), and (cv̈), respectively. As assumed by Ramsey (1927, p.52) that "λ's are linear" for the length λ_r in equation (1) or (3), he should have at least solved his problem (*ibid.*, p.55) of our equation (14) as equation (46) for n affine, linear plus intercepts, commodity markets. By the way, Ramsey (*ibid.*, p.60) conjectures that "the more complicated results $\cdots$ may well be valid under still wider conditions" than that "λ's are linear." However, our results from non-linear lengths λ_r in corollaries 4 through 6 seem to be much simpler than those with linear ones in equations (46) and (f17).

As proved in appendix B by the mathematical induction, we have shown the second-order condition for equation (14), exactly the same problem as Ramsey (1927, p.55), with a bordered Hessian $|\bar{H}|$. In addition to this, equation (B1), which is monotone decreasing as $\dfrac{d\,t}{d\,\kappa} < 0$ strictly upon a tax revenue function t of our multiplier κ or $t = t(\kappa)$ in equation (41) of proposition 2, has provided us with a necessary and sufficient condition for a unique optimal solution of an indirect taxation. We have had in the sufficient condition that not merely a slope but also a slope of its slope or a curvature of $R_r = R_r(x_r)$, an each market's tax revenue function R_r of a post-tax equilibrium quantity x_r in equation (6), has an important role on it such as a negative slope of $\dfrac{d\,R_r}{d\,x_r} \equiv R_r' < 0$ in equation (13) and a quasi-concavity of $\dfrac{d^2 R_r}{d\,x_r^2} \equiv R_r'' \leq 0$ in equation (12), so that in this way, it deserves studying if "results $\cdots$ may well be valid under still wider conditions" as

Ramsey (*ibid.*, p.60) guesses: *E. g.*, diagonal elements $\mathcal{L}_{rr}$ in equation (B1) should be negative for all $r = 1$, $2, \cdots, n$ as $\mathcal{L}_{rr} \equiv \lambda'_r - \kappa R''_r < 0$ in equation (B4) with $\lambda'_r < 0$ and $\kappa \leq 0$ in equations (8) and (A9), but it seems possible for $\mathcal{L}_{rr}$ to be negative "under still wider conditions;" how about off-diagonal elements $\mathcal{L}_{rs}$ for $r \neq s$? although we leave them open to question.

Appendixes

A. *Proofs*

In this appendix A, we prove our propositions 1 through 3 and their corollaries in order of appearance.

☛ POOF OF PROPOSITION 1:

Proof. Suppose not, or this tax rule were true: That is, we had a proportionality of $\mu_r \propto \sigma_r$ for an infinitesimal μ_r. Then, the limit of $\sigma_r = \sigma_r(x_r)$ in equation (28) must be convergent to zero as x_r goes to an initial equilibrium quantity of $\bar{x}_r$ in the set of $0 < x_r < \bar{x}_r$:

$$\text{(A1)} \quad \lim_{x_r \to \bar{x}_r^-} \sigma_r = \frac{1}{\rho_r} + \frac{1}{\varepsilon_r} = -\frac{\phi'_r(\bar{x}_r)\,\bar{x}_r}{\phi_r(\bar{x}_r)} + \frac{f'_r(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} = 0.$$

At the initial quantity of $\bar{x}_r > 0$ before tax, we have $\phi_r(\bar{x}_r) = f_r(\bar{x}_r)$, demand = supply given in equation (2). Thus, equation (A1) is reduced into

$$\frac{f'_r(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} - \frac{\phi'_r(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} = \{f'_r(\bar{x}_r) - \phi'_r(\bar{x}_r)\}\frac{\bar{x}_r}{f_r(\bar{x}_r)} = 0, \text{ or}$$

$$\text{(A2)} \quad f'_r(\bar{x}_r) = \phi'_r(\bar{x}_r),$$

which contradicts the signs of slopes: $0 \leq f'_r(\bar{x}_r) = \phi'_r(\bar{x}_r) < 0$. So, we have

$$\text{(A3)} \quad \lim_{x_r \to \bar{x}_r^-} \sigma_r(x_r) = \sigma_r(\bar{x}_r) = -\frac{\{\phi'_r(\bar{x}_r) - f'_r(\bar{x}_r)\}\,\bar{x}_r}{f_r(\bar{x}_r)} = -\frac{\lambda'_r(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} > 0$$

instead of equation (A1) by means of equation (8) or $\lambda'_r < 0$. Differently from five equations (21) through (25), the sum function $\sigma_r(x_r)$ in equation (28) is too large even at $x_r = \bar{x}_r$ to have the same order as these five equations, so that we cannot obtain the $\frac{0}{0}$ form against the function $\sigma_r = \sigma_r(x_r)$ forever.

On the other hand, it is interesting to see from the following limit of the $\frac{0}{0}$ form of equations (23) to (21) as x_r runs to $\bar{x}_r$ from the left hand side that

$$\text{(A4)} \quad \frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{\mu_r}{K_r} = \frac{\lambda_r(\bar{x}_r)}{f_r(\bar{x}_r)}\,\frac{[\{\lambda_r(x_r)\}] + \lambda'_r(\bar{x}_r)\,\bar{x}_r}{-\lambda_r(\bar{x}_r)} = \sigma_r(\bar{x}_r) \equiv c_r,$$

in a numerator of which a term denoted by $[\{\lambda_r(x_r)\}]$ of the vertical length in equation (1) or (3) should be always null as $[\{\lambda_r(x_r)\}] \approx 0$ at $x_r < \bar{x}_r$ for an infinitesimal tax rate μ_r due to equations (22) and (23). Thus, an ad valorem tax rate μ_r can be proportional to ($\propto$) his multiplier K_r or $\mu_r \propto K_r$ with a straight line of $\mu_r = c_r\,K_r$ nearby at $x_r \leq \bar{x}_r$ for an infinitesimal tax rate $\mu_r \geq 0$ where a constant coefficient c_r is given in equation (A4). □

☞ POOF OF COROLLARY 1:

Proof. As each quantity x_r may be measured by a different unit, we translate its domain of $0 < x_r \le \bar{x}_r$ in equation (2) into a unit-less quantity q_r of

$$\text{(A5)}\quad 0 < q_r \equiv \frac{x_r}{\bar{x}_r} \le 1.$$

Then, equation (A4) is able to provide us with the limit of a ratio of

$$\text{(A6)}\quad \lim_{q_r \to 1^-} \frac{\mu_i/K_i}{\mu_j/K_j} = \lim_{q_r \to 1^-} \frac{\sigma_i(\bar{x}_i\, q_i)}{\sigma_j(\bar{x}_j\, q_j)} = \frac{\sigma_i(\bar{x}_i)}{\sigma_j(\bar{x}_j)} \equiv c_{ij},$$

which can tell us even in the neighborhood of $x_r = \bar{x}_r$ or $q_r = 1$ that $\frac{\mu_i}{\mu_j} \not\propto \frac{\sigma_i(x_i)}{\sigma_j(x_j)}$ but $\frac{\mu_i}{\mu_j} \propto \frac{K_i(x_i)}{K_j(x_j)}$ at $x_r \le \bar{x}_r$ but almost near at $\bar{x}_r$ having a linear relationship as $\frac{\mu_i(\bar{x}_i\, q_i)}{\mu_j(\bar{x}_j\, q_j)} = c_{ij}\, \frac{K_i(\bar{x}_i\, q_i)}{K_j(\bar{x}_j\, q_j)}$ with a constant slope of $c_{ij} \equiv \frac{\sigma_i(\bar{x}_i)}{\sigma_j(\bar{x}_j)}$ from equation (A6) nearby at $q_r \le 1$ for all elements i, j, and $r \in \{s \mid s = 1, 2, \cdots, n\}$. □

☛ POOF OF PROPOSITION 2:

Proof. By the chain-rule, the r-th marginal market-surplus ms_r with respect to an indirect tax-payment R_r to the government, or

$$\text{(A7)}\quad \frac{d\,ms_r}{d\,R_r} = \frac{d\,ms_r}{d\,x_r}\frac{d\,x_r}{d\,R_r} = \frac{d\,ms_r}{d\,x_r}\frac{1}{d\,R_r/d\,x_r} = \frac{\lambda_r}{\lambda_r + \lambda_r'\,x_r} = \kappa \le 0$$

equals each other for all $r = 1, 2, \cdots, n$ because of equations (33), (1), and (13), and so that $\frac{d\,ms_r}{d\,x_r} = \lambda_r \ge 0$ and $\frac{d\,R_r}{d\,x_r} = \lambda_r + \lambda_r'\,x_r < 0$, respectively. □

☞ POOF OF COROLLARY 2:

Proof. The 'equal-marginal-sacrifice' of the tax-payment R_r to the government can be put into the following 'equal-elasticity-sacrifice' of that as

$$\text{(A8)}\quad \frac{d\,ms_r}{d\,R_r} = \frac{d\,ms_r}{d\,x_r}\frac{d\,x_r}{d\,R_r} = \lambda_r\frac{1}{d\,R_r/d\,x_r} = \frac{R_r/x_r}{d\,R_r/d\,x_r} = \frac{d\,x_r/x_r}{d\,R_r/R_r} = \kappa \le 0 \qquad \text{for } r = 1, 2, \cdots, n$$

by using that $\frac{d\,ms_r}{d\,x_r} = \lambda_r = \frac{R_r}{x_r}$ from equations (33) and (35), respectively. □

☛ POOF OF PROPOSITION 3:

Proof. First of all, the function t of κ, $t = t(\kappa)$ of equation (41), has a domain of $\kappa \le 0$ that is derived from equations (15), (36), and (37) as

$$\text{(A9)}\quad \kappa = \frac{\lambda_r}{\lambda_r + \lambda_r'\,x_r} = \frac{R_r/x_r}{d\,R_r/d\,x_r} = \frac{R_r/q_r}{d\,R_r/d\,q_r} \le 0$$

due to a non-negative numerator of the vertical length $\lambda_r \ge 0$ in equation (1) or (3), and due to a negative denominator of the marginal tax revenue $\frac{d\,R_r}{d\,x_r} < 0$ in equation (13). It is trivial for all $r = 1, 2, \cdots, n$ that $\lambda_r = \kappa = t = 0$. Next, for a certain amount of $R_r^{\exists}$ less than the r-th market's 'ability' of maximum tax-payment, say R_r^{max}: $0 \le R_r^{\exists} < R_r^{max}$; in total, we have $0 \le R \equiv \sum_{r=1}^{n} R_r^{\exists} < \bar{R} \equiv \sum_{r=1}^{n} R_r^{max}$. Therefore, equation (41), the function $t = t(\kappa)$, has a region of $0 \leq t = R < \bar{R}$. By employing the chain rule, at last, it is easy to see in the domain $\kappa \leq 0$ that the tax revenue function $t = t(\kappa)$ is monotone decreasing with respect to the multiplier κ as

$$\text{(A10)} \quad \frac{d\,t}{d\,\kappa} = \sum_{r=1}^{n} \frac{d\,R_r}{d\,x_r} \frac{d\,x_r}{d\,\kappa} = \sum_{r=1}^{n} \frac{d\,R_r}{d\,x_r} \frac{1}{d\,\kappa / d\,x_r} < 0$$

because of not only the marginal tax revenue of equation (13) as $\dfrac{d\,R_r}{d\,x_r} < 0$ but also the following marginal multiplier of equation (15) as $\dfrac{d\,\kappa}{d\,x_r} > 0$, or

$$\frac{d\,\kappa}{d\,x_r} = \frac{\lambda_r' \, (\lambda_r + \lambda_r' \, x_r) - \lambda_r \, (2\,\lambda_r' + \lambda_r'' \, x_r)}{(\lambda_r + \lambda_r' \, x_r)^2} = \frac{\lambda_r' \, (d\,R_r / d\,x_r) - \lambda_r \, (d^2\,R_r / d\,x_r^2)}{(d\,R_r / d\,x_r)^2} > 0$$

owing to equations (1) or (3), (8), (12), and (13) as $\lambda_r \geq 0$, $\lambda_r' < 0$, $\dfrac{d^2\,R_r}{d\,x_r^2} \leq 0$, and $\dfrac{d\,R_r}{d\,x_r} < 0$, respectively, some of which simultaneously make sure of the second-order condition according to equations (B3) and (B4) in appendix B. Consequently, such monotone decreasing of $\dfrac{d\,t}{d\,\kappa} < 0$ guarantees us a unique optimal solution κ^* for $t = R$ somewhere in its domain $\kappa \leq 0$. □

☞ POOF OF COROLLARY 3:

Proof. As developed in section Ⅱ.A, we employ the third procedure of equation (37) or (A9) with the unit-less quantity q_r of equation (A5) as

$$\text{(37)} \quad \kappa = \frac{R_r / q_r}{d\,R_r / d\,q_r} \leq 0; \;\; 0 < q_r \equiv \frac{x_r}{\bar{x}_r} \leq 1 \qquad \text{for } r = 1, 2, \cdots, n.$$

First of all, equalizing equations (42) and (43) of $-\,\alpha_r\,x_r + \beta_r = a_r\,x_r + b_r$ gives us each market's initial quantity $\bar{x}_r$ before tax as $\bar{x}_r^B \equiv \dfrac{\beta_r - b_r}{\alpha_r + a_r}$. Then, we can put not only a quantity variable x_r in the domain of equation (2) or $0 < x_r \leq \bar{x}_r$ into equation (A5) as $x_r = \bar{x}_r^B\,q_r$ but also a tax revenue variable R_r of equation (6) or $R_r = (-\,\alpha_r\,x_r + \beta_r - a_r\,x_r - b_r)\,x_r$ in this case into

$$\text{(A11)} \quad R_r^B \equiv -\,4\,\hat{R}_r^{max}\,q_r^2 + 4\,\hat{R}_r^{max}\,q_r = -\,4\,\hat{R}_r^{max}\,q_r\,(q_r - 1)$$

where $\hat{R}_r^{max}$ is Hat-type 'ability' of a maximum amount in equation (8), which every market can potentially pay for an indirect tax to a government; as $q_r \to 0.5^+$ in all markets, it approaches $\hat{R}_r^{max} = \dfrac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$. So, our equation (37) conveys equation (A11) into

$$\text{(A12)} \quad \kappa = \kappa(q_r) \equiv \frac{R_r^B / q_r}{d\,R_r^B / d\,q_r} = \frac{q_r - 1}{2\,q_r - 1} \leq 0; \;\; 0.5 < q_r \equiv \frac{x_r}{\bar{x}_r^B} \leq 1$$

due to an average function $\dfrac{R_r^B}{q_r} = -\,4\,\hat{R}_r^{max}\,(q_r - 1)$ and the derivative $\dfrac{d\,R_r^B}{d\,q_r} = -\,4\,\hat{R}_r^{max}\,(2q_r - 1)$. And its inverse function q_r is provided as

$$\text{(A13)} \quad q_r = q_r(\kappa) \equiv \frac{\kappa - 1}{2\,\kappa - 1}; \;\; \kappa \leq 0.$$

By using equation (41) in proposition 3 or equation (40) for this proof, next, a sum of equation (A11) with equation (A13) for a tax revenue function t of κ equal to R, the government's constraint for her tax revenue, is given as

$$\text{(A14)}\quad t = t(\kappa) \equiv \sum_{r=1}^{n} -4\,\hat{R}_r^{max}\, q_r\,(q_r - 1) = -4\,\hat{R}\,\frac{\kappa - 1}{2\,\kappa - 1}\,\frac{-\kappa}{2\,\kappa - 1} = R$$

where $\hat{R}$ is the total sum as $\hat{R} \equiv \sum_{r=1}^{n} \hat{R}_r^{max} = \sum_{r=1}^{n} \frac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$; $0 \leq R < \hat{R}$. Equation (A14) is nothing but equation (44) itself, which yields the following quadratic function to us:

$$\text{(A15)}\quad (\hat{R} - R)\,\kappa^2 - (\hat{R} - R)\,\kappa - \frac{R}{4} = 0;\;\; \kappa \leq 0,\;\; 0 \leq R < \hat{R}.$$

Its discriminant, say D is always positive as $D \equiv (\hat{R} - R)^2 + (\hat{R} - R)\,R = (\hat{R} - R)\,\hat{R} > 0$ so that its quadratic formula produces two roots of

$$\text{(A16)}\quad \kappa = \frac{\hat{R} - R \pm \sqrt{(\hat{R} - R)\,\hat{R}}}{2\,(\hat{R} - R)}$$

whose negative root directly becomes equation (45) owing to the domain of $\kappa \leq 0$. Finally, substituting equation (45) back into equation (A13) as

$$\text{(A17)}\quad q_r = \frac{\kappa^* - 1}{2\,\kappa^* - 1} = \frac{\hat{R} - R + \sqrt{(\hat{R} - R)\,\hat{R}}}{2\,\sqrt{(\hat{R} - R)\,\hat{R}}} = \frac{\hat{R} + \sqrt{(\hat{R} - R)\,\hat{R}}}{2\,\hat{R}}$$

and plugging its substituted equation (A17) back into equation (A11) sequentially provides us with equation (47) immediately. On the other hand, plugging the above equation (A17) back into equation (4) with $x_r = \bar{x}_r^B\, q_r$ as

$$\text{(A18)}\quad \mu_r = \frac{\phi_r(x_r) - f_r(x_r)}{f_r(x_r)} = \frac{-\alpha_r\,\bar{x}_r^B\, q_r + \beta_r - a_r\,\bar{x}_r^B\, q_r - b_r}{a_r\,\bar{x}_r^B\, q_r + b_r},$$

one has equation (46) at once where $\bar{x}_r^B$ is the initial quantity $\bar{x}_r$ before tax of $\bar{x}_r^B \equiv \frac{\beta_r - b_r}{\alpha_r + a_r}$ as discussed earlier in this corollary. □

☞ POOF OF COROLLARIES 4 and 5:

Proof. Similarly to the previous proof, first of all, setting equation (48) equal to equations (49) and (54) as $\frac{\gamma_r}{x_r} = a_r\,x_r$ and as $\frac{\gamma_r}{x_r} = b_r$, respectively, gives us each market's initial quantity $\bar{x}_r$ before tax of $\bar{x}_r^C \equiv \sqrt{\gamma_r / a_r}$ and of $\bar{x}_r^H \equiv \frac{\gamma_r}{b_r}$. Then, we can transfer not only a quantity variable x_r in the domain of equation (2) or $0 < x_r \leq \bar{x}_r$ into equation (A5) as $x_r = \bar{x}_r^C\, q_r$ as well as $x_r = \bar{x}_r^H\, q_r$ but also a tax revenue variable R_r of equation (6) or $R_r^C \equiv (\frac{\gamma_r}{x_r} - a_r\,x_r)\,x_r$ and $R_r^H \equiv (\frac{\gamma_r}{x_r} - b_r)\,x_r$ here into

$$\text{(A19)}\quad R_r^C = \grave{R}_r^{max}\,(1 - q_r^2),$$

in which $\grave{R}_r^{max}$ is Grave-type 'ability' of a maximum amount in equation (10) that a market can potentially pay for an indirect tax to a government: $I.e.$, it arrives at $\grave{R}_r^{max} = \gamma_r$ as $q_r \to 0^+$ in every market; and into

(A20) $R_r^H = \grave{R}_r^{max}(1 - q_r)$,

respectively. So, our equation (37) puts equations (A19) and (A20) into

(A21) $\kappa = \kappa(q_r) \equiv \dfrac{R_r^C/q_r}{d\,R_r^C/d\,q_r} = \dfrac{q_r^2 - 1}{2\,q_r^2} \leq 0;\ 0 < q_r \equiv \dfrac{x_r}{\bar{x}_r^C} \leq 1$

because of $\dfrac{R_r^C}{q_r} = \dfrac{\grave{R}_r^{max}(1 - q_r^2)}{q_r}$ as well as $\dfrac{d\,R_r^C}{d\,q_r} = -\,2\,\grave{R}_r^{max}\,q_r$; and into

(A22) $\kappa = \kappa(q_r) \equiv \dfrac{R_r^H/q_r}{d\,R_r^H/d\,q_r} = \dfrac{q_r - 1}{q_r} \leq 0;\ 0 < q_r \equiv \dfrac{x_r}{\bar{x}_r^H} \leq 1$

owing to $\dfrac{R_r^H}{q_r} = \dfrac{\grave{R}_r^{max}(1 - q_r)}{q_r}$ and the derivative $\dfrac{d\,R_r^H}{d\,q_r} = -\,\grave{R}_r^{max}$. Then, their inverse functions for equations (A21) and (A22) are calculated as

(A23) $q_r = q_r(\kappa) \equiv \sqrt{\dfrac{1}{1 - 2\,\kappa}};\ \kappa \leq 0,$

(A24) $q_r = q_r(\kappa) \equiv \dfrac{1}{1 - \kappa};\ \kappa \leq 0,$

respectively. A sum of equation (A19) with equation (A23) for a function t of κ equal to R, the government's tax revenue, is given as equation (51) of

(A25) $t = t(\kappa) \equiv \sum_{r=1}^{n} \grave{R}_r^{max}(1 - q_r^2) = \grave{R}\,\dfrac{2\kappa}{2\,\kappa - 1} = R$

where the total sum $\grave{R} \equiv \sum_{r=1}^{n} \grave{R}_r^{max} = \sum_{r=1}^{n} \gamma_r > R \geq 0$. In a similar manner,

(A26) $t = t(\kappa) \equiv \sum_{r=1}^{n} \grave{R}_r^{max}(1 - q_r) = \grave{R}\,\dfrac{\kappa}{\kappa - 1} = R$

due to a sum of equation (A20) with equation (A24). Solving equation (A25) of $2\,\kappa\,\grave{R} = (2\,\kappa\,-\,1)\,R$ with respect to κ (automatically non-positive: $\kappa \leq 0$) yields equation (50). Similarly, equation (A26) of $\kappa\,\grave{R} = (\kappa\,-\,1)\,R$ turns to be an equation $\bar{\kappa}^{**}$ (≤ 0) in corollary 5. Next, substituting equations (50) and the equation $\bar{\kappa}^{**}$ back into equations (A23) and (A24), respectively, as

(A27) $q_r = \sqrt{\dfrac{-1}{2\,\kappa^{**} - 1}} = \sqrt{\dfrac{-1}{-2\,R/\{2\,(\grave{R} - R)\} - 1}} = \dfrac{\sqrt{(\grave{R} - R)\,\grave{R}}}{\grave{R}},$

(A28) $q_r = \dfrac{1}{1 -\ \bar{\kappa}^{**}} = \dfrac{1}{1 + R/(\grave{R} - R)} = \dfrac{\grave{R} - R}{\grave{R}}.$

By plugging substituted equations (A27) and (A28) back into equation (A19) and (A20), respectively, one has exactly the same equation (53). Finally, by substituting equations (A27) and (A28) into equation (4) with $x_r = \bar{x}_r^C\,q_r$ and with $x_r = \bar{x}_r^H\,q_r$ where $\bar{x}_r^C \equiv \sqrt{\gamma_r/a_r}$ and $\bar{x}_r^H \equiv \dfrac{\gamma_r}{b_r}$ as follows:

(A29) $\mu_r = \dfrac{\phi_r(x_r) - f_r(x_r)}{f_r(x_r)} = \dfrac{\gamma_r/(\bar{x}_r^C\,q_r) - a_r\,\bar{x}_r^C\,q_r}{a_r\,\bar{x}_r^C\,q_r};$

(A30) $\mu_r = \dfrac{\phi_r(x_r) - f_r(x_r)}{f_r(x_r)} = \dfrac{\gamma_r/(\bar{x}_r^H\, q_r) - b_r}{b_r}$,

one has nothing but equations (52) and (56), respectively. □

☞ POOF OF COROLLARY 6:

Proof. From equations (A14) and (A25), each tax revenue function of κ can be written as $4\,\hat{R}_1^{max}\,\dfrac{\kappa(\kappa-1)}{(2\,\kappa-1)^2}$, $\grave{R}_2^{max}\,\dfrac{2\kappa}{2\,\kappa-1}$, and $\grave{R}_3^{max}\,\dfrac{2\kappa}{2\,\kappa-1}$, in which $\hat{R}_1^{max} = \dfrac{(\beta_1 - b_1)^2}{4\,(\alpha_1 + a_1)}$, $\grave{R}_2^{max} = \gamma_2$, and $\grave{R}_3^{max} = \gamma_3$, respectively, as shown in equations (8) and (10). Those tax revenue functions of κ add up to equation (58), so that by setting it equal to a government's tax revenue R, we can have the following quadratic function of κ as

(A31) $4\,(\hat{R}_{mix}^{max} - R)\,\kappa^2 - 2\,(\hat{R}_{mix}^{max} + \hat{R}_1^{max} - 2\,R)\,\kappa - R = 0$

where $\hat{R}_{mix}^{max} \equiv \hat{R}_1^{max} + \grave{R}_2^{max} + \grave{R}_3^{max} = \dfrac{1}{4}\,\dfrac{(\beta_1 - b_1)^2}{\alpha_1 + a_1} + \gamma_2 + \gamma_3$. Due to a positive discriminant $\dfrac{D}{4} \equiv (\hat{R}_{mix}^{max} + \hat{R}_1^{max} - 2\,R)^2 + 4\,(\hat{R}_{mix}^{max} - R)\,R > 0$, in which $0 \leq R < \hat{R}_{mix}^{max}$, a negative real root in equation (A31) becomes equation (59) itself owing to the domain of $\kappa \leq 0$. □

B. *A Bordered Hessian*

In this appendix, we discuss the second-order condition for the Lagrange function $\mathcal{L}$ in equation (14) as well as a bordered Hessian $|\bar{H}|$ and its principal minors, $|\bar{H}_2|$, $|\bar{H}_3|$, $\cdots$, and $|\bar{H}_n|$ with the last one being that $|\bar{H}_n| = |\bar{H}|$.

Denote by g a sum of tax revenues R_r in equation (6) for $r = 1, 2, \cdots, n$ or $g \equiv \sum_{r=1}^{n} \lambda_r\, x_r$, and rewrite the Lagrange function $\mathcal{L}$ in equation (14) as

(14') $\mathcal{L} \equiv -\sum_{r=1}^{n} \int_{x_r}^{\bar{x}_r} \lambda_r\, ds_r + \kappa\,(R - g)$

where κ is the multiplier, then we have a bordered Hessian $|\bar{H}|$ of

(B1) $$|\bar{H}| \equiv \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & \mathcal{L}_{11} & 0 & \cdots & 0 \\ g_2 & 0 & \mathcal{L}_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & 0 & 0 & \cdots & \mathcal{L}_{nn} \end{vmatrix}$$

including its successive bordered principal minors as

(B2) $$|\bar{H}_2| \equiv \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & \mathcal{L}_{11} & 0 \\ g_2 & 0 & \mathcal{L}_{22} \end{vmatrix}, |\bar{H}_3| \equiv \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & \mathcal{L}_{11} & 0 & 0 \\ g_2 & 0 & \mathcal{L}_{22} & 0 \\ g_3 & 0 & 0 & \mathcal{L}_{33} \end{vmatrix}, \cdots, |\bar{H}_n|,$$

the last of which must be exactly the same as equation (B1) or $|\bar{H}_n| \equiv |\bar{H}|$ where all non-zero components of partial derivatives g_r and $\mathcal{L}_{rr}$ are supposed to be negative with $\dfrac{d\,R_r}{d\,x_r} \equiv R_r' < 0$ in equation (13) and with

$\lambda_r' < 0$, $\kappa \le 0$, and $\dfrac{d^2 R_r}{d\,x_r^2} \equiv R_r'' \le 0$ in equations (8), (A9), and (12), respectively, as

$$\text{(B3)}\quad g_r \equiv \frac{\partial R_r}{\partial x_r} = \frac{d}{d\,x_r}(\lambda_r\, x_r) = R_r' = \lambda_r'\, x_r + \lambda_r < 0,$$

$$\text{(B4)}\quad \mathcal{L}_{rr} \equiv \frac{\partial^2 \mathcal{L}}{\partial x_r^2} = \frac{d}{d\,x_r}(\frac{\partial \mathcal{L}}{\partial x_r}) = \frac{d}{d\,x_r}\{\lambda_r - \kappa\,(\lambda_r'\, x_r + \lambda_r)\}$$
$$= \lambda_r' - \kappa\,(\lambda_r''\, x_r + 2\,\lambda_r') = \lambda_r' - \kappa\, R_r'' < 0.$$

Therefore, principal minors in equation (B2) alternately become positive and negative as $|\bar{H}_2| = -\,\mathcal{L}_{22}\, g_1^2 -\mathcal{L}_{11}\, g_2^2 > 0$, $|\bar{H}_3| = \mathcal{L}_{33}\,|\bar{H}_2| - \mathcal{L}_{11}\,\mathcal{L}_{22}\, g_3^2 < 0$, $|\bar{H}_4| = \mathcal{L}_{44}\,|\bar{H}_3| - \mathcal{L}_{11}\,\mathcal{L}_{22}\,\mathcal{L}_{33}\, g_4^2 > 0$, $\cdots$, and so provided that $|\bar{H}_n| = (-1)^n |\bar{H}_n| > 0$ as Alpha C. Chiang (1984, p.385), then we are always able to induce further in the Ramsey's tax problem like equation (14') here that $|\bar{H}_{n+1}| = \mathcal{L}_{n+1}\,|\bar{H}_n| - \mathcal{L}_{11}\,\mathcal{L}_{22}\cdots\mathcal{L}_{nn}\, g_{n+1}^2 < 0$ due to equations (B3) and (B4): $g_r < 0$; $\mathcal{L}_{rr} < 0$ for $r = 1, 2, \cdots, n$, and $n+1$.

Then, all we have to do for the second-order condition corresponding to an optimal indirect tax is to check out whether or not all non-zero elements of the bordered Hessian $|\bar{H}|$ in equation (B1), or partial derivatives g_r and $\mathcal{L}_{rr}$ are negative: $g_r < 0$; $\mathcal{L}_{rr} < 0$ for $r = 1, 2, \cdots$, and n.

So, it is a piece of cake to observe in the affine n-commodity markets of corollary 3 that every non-zero element of $g_r = R_r'$ or $\mathcal{L}_{rr} = \lambda_r' - \kappa\, R_r''$ is negative as follows: ✍❶ Because equations (42) and (43) give us the vertical length λ_r in equation (1) or (3) as $\lambda_r \equiv -(\alpha_r + a_r)\, x_r + \beta_r - b_r$ in the domain of $0 < x_r \le \bar{x}_r^B \equiv (\beta_r - b_r)/(\alpha_r + a_r)$ from equation (2), its derivative λ_r' and the second one λ_r'' are calculated as $\lambda_r' = -(\alpha_r + a_r) < 0$ and $\lambda_r'' = 0$, respectively. ✍❷ Those equations also give us the tax revenue of R_r in equation (6) as $R_r = \lambda_r\, x_r = -(\alpha_r + a_r)\, x_r^2 + (\beta_r - b_r)\, x_r$ so that its derivative of $R_r' = \lambda_r'\, x_r + \lambda_r$ and the second one of $R_r'' = \lambda_r''\, x_r + 2\,\lambda_r'$ can be obtained as $R_r' = -2\,(\alpha_r + a_r)\, x_r + \beta_r - b_r < 0$ as long as $0.5\,\bar{x}_r^B < x_r \le \bar{x}_r^B$ or $0.5 < q_r \equiv x_r/\bar{x}_r^B \le 1$ in terms of the unit-less quantity q_r in equation (A5), but as $R_r'' = 2\,\lambda_r' < 0$ all the time. ✍❸ Recall in equation (A12) that the multiplier κ keeps its domain of $\kappa \le 0$ as long as $0.5 < q_r \equiv x_r/\bar{x}_r^B \le 1$, and so that let x_r^* be an equilibrium quantity corresponding to an optimal indirect tax or $0.5 < q_r \equiv x_r^*/\bar{x}_r^B \le 1$, then we always have $g_r = R_r' < 0$, $\kappa \le 0$, and $\mathcal{L}_{rr} = (1 - 2\,\kappa)\,\lambda_r' < 0$ for all $r = 1, 2, \cdots, n$. In fact, we have shown an optimal κ as $\kappa^* = \{\hat{R} - R - \sqrt{(\hat{R} - R)\,\hat{R}}\}/\{2\,(\hat{R} - R)\} \le 0$ in equation (45).

Similarly, in the n-commodity markets of corollary 4, ✍❹ since equations (48) and (49) yield the vertical length of λ_r in equation (1) or (3) as $\lambda_r \equiv \gamma_r/x_r - a_r\, x_r$ in the domain of $0 < x_r \le \bar{x}_r^C \equiv \sqrt{\gamma_r/a_r}$ from equation (2), its derivative λ_r' and the second one λ_r'' are computed as $\lambda_r' = -\gamma_r/x_r^2 - a_r < 0$ and $\lambda_r'' = 2\,\gamma_r/x_r^3 > 0$. ✍❺ The tax revenue of R_r in equation (6) is given as $R_r = \lambda_r\, x_r = \gamma_r - a_r\, x_r^2$ so that its derivative R_r' and the second one R_r'' can be computed as $R_r' = -2\, a_r\, x_r < 0$ and $R_r'' = \lambda_r''\, x_r + 2\,\lambda_r' = -2\, a_r < 0$. ✍❻ Recall in equation (A21) that κ has a domain of $\kappa \le 0$ with a region of the unit-less q_r as $0 < q_r \equiv x_r^{**}/\bar{x}_r^C \le 1$ for an optimal equilibrium quantity after tax x_r^{**}, then one always has $g_r = R_r' < 0$, $\kappa \le 0$, and $\mathcal{L}_{rr} = \lambda_r' - \kappa\, R_r'' = \lambda_r' + 2\, a_r\,\kappa < 0$ for all $r = 1, 2, \cdots$, and n. Actually, she or he has had an optimal κ as $\kappa^{**} = -R/\{2\,(\grave{R} - R)\} \le 0$ in equation (50).

Moreover, in the n-commodity markets of corollary 5, ✍❼ since equations (48) and (54) give us the vertical length λ_r in equation (1) or (3) as $\lambda_r \equiv \gamma_r/x_r - b_r$ in the domain of $0 < x_r \le \bar{x}_r^H \equiv \gamma_r/b_r$ from equation (2), its derivative λ'_r and the second one λ''_r are computed as $\lambda'_r = -\gamma_r/x_r^2 < 0$ and $\lambda''_r = 2\,\gamma_r/x_r^3 > 0$. ✍❽ The tax revenue of R_r in equation (6) is given as $R_r = \lambda_r\, x_r = \gamma_r - b_r\, x_r$ so that its derivative R'_r and the second one R''_r can be calculated as $R'_r = -\,b_r < 0$ and $R''_r = \lambda''_r\, x_r + 2\,\lambda'_r = 0$. ✍❾ Recall in equation (A22) that κ has a domain of $\kappa \leq 0$ with a region of the unit-less q_r as $0 < q_r \equiv \bar{x}_r^{**}/\bar{x}_r^H \leq 1$ for an optimal equilibrium quantity after tax $\bar{x}_r^{**}$, then we always have that $g_r = R'_r < 0$, $\kappa \leq 0$, and that $\mathcal{L}_{rr} = \lambda'_r - \kappa\, R''_r = \lambda'_r < 0$ for all $r = 1, 2, \cdots$, and n, in which we have already had a non-positive optimal κ as $\bar{\kappa}^{**} = -R/(\grave{R} - R) \leq 0$ in corollary 5.

Furthermore, it can be observed in corollary 6 with parameters $a_3 = \dfrac{1}{3}$, $\alpha_1 = \gamma_2 = a_1 = b_1 = 1$, $\gamma_3 = 3$, $a_2 = 4$, and $\beta_1 = 5$ reflecting four optimal choice variables, the multiplier $\tilde{\kappa}^*_{mix} = -0.5$ in equation (60), equilibrium quantities after tax $\tilde{x}_1^* = 1.5$, $\tilde{x}_2^* = \sqrt{2}/4$, and $\tilde{x}_3^* = 3\sqrt{2}/2$ that all non-zero elements in a bordered Hessian are negative as follows: ✌① As shown in ✍❶ and ✍❹, the derivatives λ'_r are computed as $\lambda'_1 = -(\alpha_1 + a_1) = -2 < 0$, $\lambda'_2 = -\gamma_2/(\tilde{x}_2^*)^2 - a_2 = -12 < 0$, and $\lambda'_3 = -\gamma_3/(\tilde{x}_3^*)^2 - a_3 = -1 < 0$, respectively. ✌② Besides, as shown in ✍❷ and ✍❺, the derivatives R'_r are calculated as $R'_1 = -2\,(\alpha_1 + a_1)\,\tilde{x}_1^* + \beta_1 - b_1 = -2 < 0$, $R'_2 = -2\,a_2\,\tilde{x}_2^* = -2\sqrt{2} < 0$, and $R'_3 = -2\,a_3\,\tilde{x}_3^* = -\sqrt{2} < 0$, respectively; on the other hand, the second derivatives R''_r are obtained as $R''_1 = -2\,(\alpha_1 + a_1) = -4 < 0$, $R''_2 = -2\,a_2 = -8 < 0$, and $R''_3 = -2\,a_3 = -2/3 < 0$, respectively. ✌③ As seen in ✍❸ and ✍❻, therefore, all non-zero elements $g_r = R'_r$ as well as $\mathcal{L}_{rr} = \lambda'_r - \tilde{\kappa}^*_{mix}\, R''_r$ are strictly negative as $g_1 = -2 < 0$, $g_2 = -2\sqrt{2} < 0$, $g_3 = -\sqrt{2} < 0$, $\mathcal{L}_{11} = -4 < 0$, $\mathcal{L}_{22} = -16 < 0$, and $\mathcal{L}_{33} = -4/3 < 0$, respectively.

C. *Equations (3) and (11) in Frank P. Ramsey*

In this appendix, we review equations (3) and (11) (where we underline to let them different from ours) in Ramsey (1927, p.49, p.56). To begin with, recall in equation (6) and in figure 1 that the government's tax revenue of

$$(6) \quad R_r \equiv \lambda_r\, x_r = \{\phi_r(x_r) - f_r(x_r)\}\, x_r$$

is geometrically described by a rectangular area or a product of the vertical length λ_r in equation (1) or (3) times the width of the quantity x_r for $r = 1, 2, \cdots, n$. In each market, according to equations (7) and (9), the maximum tax revenue R_r^{max} of equation (6) ought to retain equation (11) or

$$(\text{C1}) \quad \frac{d\,R_r}{d\,x_r} \equiv \lambda'_r\, x_r + \lambda_r \leq 0$$

at $x_r = x_r^{max}$; this equal sign in equation (C1) or equation (7) gives us

$$(\text{C2}) \quad \frac{-\lambda_r}{\lambda'_r\, x_r} = 1.$$

Now, denote by θ_r an elasticity of width x_r with respect to the length λ_r, *i.e.*, the length elasticity of width in the geometrical rectangle reflecting the tax revenue R_r in equation (6) as shown in figure 1 as

$$(\text{C3}) \quad \theta_r \equiv \frac{-\lambda_r}{\lambda'_r\, x_r} = \frac{-\{\phi_r(x_r) - f_r(x_r)\}}{\{\phi'_r(x_r) - f'_r(x_r)\}\, x_r},$$

then equations (C1), (C2), (5), and (8) yield the following regions to us:

$$\text{(C4)}\quad \frac{d\,R_r}{d\,\mu_r} = \lambda'_r\,x_r(1-\theta_r)\,\frac{1}{\mu'_r} = \begin{cases} \geq 0 & \text{if } 0 \leq \theta_r < 1; \\ = 0 & \text{if } \theta_r = 1 \text{ with } R_r^{max} = \hat{R}_r^{max}; \\ < 0 & \text{if } \theta_r > 1; \end{cases}$$

owing to the chain rule of equations (C1) and (5):

$$\frac{d\,R_r}{d\,\mu_r} = \frac{d\,R_r}{d\,x_r}\,\frac{d\,x_r}{d\,\mu_r} = (\lambda'_r\,x_r + \lambda_r)\,\frac{d\,x_r}{d\,\mu_r} = \lambda'_r\,x_r(1 - \frac{-\lambda_r}{\lambda'_r\,x_r})\,\frac{1}{\mu'_r}.$$

As discussed in footnote 12, equation (3) in Ramsey (1927, p.49) is our equation (C3), but it should have been identical to

$$\text{(C5)}\quad \frac{\partial\,u_r/\partial\,x_r}{\partial\,R_r/\partial\,x_r} = \frac{\lambda_r}{\lambda_r + \lambda'_r\,x_r} = -K$$

for $r = 1, 2, \cdots, n$. So, it is easy to show that the multiplier K turns to be

$$\text{(C6)}\quad K = -\frac{\lambda_r/(\lambda'_r\,x_r)}{\lambda_r/(\lambda'_r\,x_r)+1} = \frac{\theta_r}{1-\theta_r}$$

in terms of the elasticity $\theta_r \equiv \dfrac{-\lambda_r}{\lambda'_r\,x_r}$ in equation (C3). Since equation (C6) holds for all r and $s \in \{r \mid r = 1, 2, \cdots, n\}$, one might ignore or eliminate the multiplier K and set $\dfrac{\theta_r}{1-\theta_r} = \dfrac{\theta_s}{1-\theta_s}$ to obtain $\theta_r\,(1-\theta_s) = \theta_s\,(1-\theta_r)$. As seen in footnote 6, equations (C3) and (C6) here, there may be three ways to redundantly take $\theta = \theta_r = \theta_s$ as a kind of new multiplier like

$$\text{(C7)}\quad K = \frac{\theta}{1-\theta} = \begin{cases} \geq 0 & \text{if } 0 \leq \theta < 1, \\ = \pm\infty & \text{if } \theta = 1 \text{ with } R_r^{max} = \hat{R}_r^{max}, \\ < 0 & \text{if } \theta > 1, \end{cases}$$

in which the multiplier K has a jump region at $\theta = 1$. It is easy to calculate its derivative with respect to θ or $\dfrac{d\,K}{d\,\theta} = \dfrac{1}{(1-\theta)^2} > 0$ everywhere at $\theta \neq 1$.

Next, it can be shown from equations (1) or (3), (4), (C3), and (C7) that

$$\text{(C8)}\quad \mu_r = \frac{\lambda_r}{f_r} = \frac{-\lambda_r}{\lambda'_r\,x_r}\,\frac{-\lambda'_r\,x_r}{f_r} = \theta_r\,\frac{-\lambda'_r\,x_r}{f_r} = \theta\,\frac{-\lambda'_r\,x_r}{f_r},$$

and that equation (C8) becomes equation (11) in Frank P. Ramsey (1927) as

$$\text{(C9)}\quad \mu_r = \frac{\theta(1/\varepsilon_r + 1/\rho_r)}{1-\theta/\rho_r}$$

in terms of three elasticities: ρ_r and ε_r as well as θ; *i.e.*, two price elasticities of demand $\rho_r \equiv -\dfrac{\phi_r}{\phi'_r\,x_r}$ and supply $\varepsilon_r \equiv \dfrac{f_r}{f'_r\,x_r}$ in equation (28) as well as the length elasticity of width $\theta = \theta_r \equiv \dfrac{-\lambda_r}{\lambda'_r\,x_r}$ in equation (C3), respectively, because we are able to put equation (C8) into $\mu_r = \theta\ (f'_r\,x_r/f_r - \phi'_r\,x_r/f_r) = \theta\ [f'_r\,x_r/f_r - \phi'_r\,x_r/\{\phi_r/(1+\mu_r)\}] = \theta\ [1/(f_r/f'_r\,x_r) + (1+\mu_r)/\{-\phi_r/(\phi'_r\,x_r)\}] = \theta\ \{1/\varepsilon_r + (1+\mu_r)\,(1/\rho_r)\} = \theta\ (1/\varepsilon_r + 1/\rho_r) + \mu_r\,\theta/\rho_r$ due to $\phi_r = (1+\mu_r)\,f_r$ from equation (3).

So, other things being equal in equation (C9), Ramsey (1927, p.56) unusually treats only θ as a running parameter in taking its limit as θ approaches zero from the right (0^+) to claim a proportionality ($\propto$) of

$$\text{(C10)}\quad \mu_r \propto \left(\frac{1}{\varepsilon_r} + \frac{1}{\rho_r}\right) \equiv \sigma_r$$

although his unusual limit does not seem to result in equation (C10) but in

$$\text{(C11)}\quad \mu_r = \frac{\theta(1/\varepsilon_r + 1/\rho_r)}{1 - \theta/\rho_r} \to 0 \quad \text{as } \theta \to 0^+.$$

Therefore, it is quite apparent as proved in proposition 1 such an infinitesimal ad valorem tax rate μ_r could not have been proportional to the sum function σ_r in equation (28) at all as

$$\text{(C12)}\quad \mu_r \not\propto \left(\frac{1}{\varepsilon_r} + \frac{1}{\rho_r}\right) \equiv \sigma_r$$

everywhere in the domain of equation (2) or $0 < x_r \leq \bar{x}_r$. It is also quite apparent from equation (C9) that the ad valorem tax rate $\mu_r \propto \theta$, the length elasticity of width.

Equation (C9) just provides us with another $\frac{0}{0}$ form, a part of which is well known as the l'Hôpital's rule such as

$$\text{(l'Hôpital)}\quad \frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{\mu_r}{\theta} = \lim_{x_r \to \bar{x}_r^-} \frac{\{\mu_r(x_r) - \mu_r(\bar{x}_r)\}/(x_r - \bar{x}_r)}{\{\theta(x_r) - \theta_r(\bar{x}_r)\}/(x_r - \bar{x}_r)} = \frac{\mu_r'(\bar{x}_r)}{\theta'(\bar{x}_r)}$$

because of $\mu_r(\bar{x}_r) = \theta(\bar{x}_r) = 0$ from equations (4) and (C3), and so that

$$\text{(C13)}\quad \frac{\mu_r'(\bar{x}_r)}{\theta'(\bar{x}_r)} = \left.\frac{d\,\mu_r/d\,x_r}{d\,\theta/d\,x_r}\right|_{x_r=\bar{x}_r} = \left.\frac{d\,\mu_r}{d\,\theta}\right|_{x_r=\bar{x}_r} = \left.\frac{d\,\mu_r}{d\,\theta}\right|_{\mu_r=\theta_r=0}.$$

In fact, the $\frac{0}{0}$ form of equation (l'Hôpital) or (C13) becomes

$$\text{(C14)}\quad \frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{\mu_r}{\theta} = \lim_{x_r \to \bar{x}_r^-} \frac{\lambda_r(x_r)/f_r(x_r)}{-\lambda_r(x_r)/\{\lambda_r'(x_r)\,x_r\}} = -\lambda_r'(\bar{x}_r)\frac{\bar{x}_r}{f(\bar{x}_r)},$$

then the fact of $f_r(\bar{x}_r) = \phi_r(\bar{x}_r)$ at $x_r = \bar{x}_r$ in equation (1) or (3) brings us

$$-\lambda_r'(\bar{x}_r)\frac{\bar{x}_r}{f(\bar{x}_r)} = \frac{f_r'(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} - \frac{\phi_r'(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} = \frac{f_r'(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} - \frac{\phi_r'(\bar{x}_r)\,\bar{x}_r}{\phi_r(\bar{x}_r)} = \sigma_r(\bar{x}_r) > 0.$$

As discussed in equation (A4) that the term $[\{\lambda_r(x_r)\}]$ should be null even for an infinitesimal rate μ_r, it seems worth reminding here that in a denominator of equation (C14), one needs $[\{\lambda_r(x_r)\}] = \phi_r(x_r) - f_r(x_r) \approx 0$ at $x_r < \bar{x}_r$ as a proxy for the fact of $f_r(\bar{x}_r) = \phi_r(\bar{x}_r)$ at $x_r = \bar{x}_r$ in equation (1) or (3).

At last, hence, the limit of the $\frac{0}{0}$ form from equation (C9) gives us exactly the same result from proposition 1 or equations (A3) and (A4) as well as one from the above equation (C14) as follows:

$$\text{(C15)}\quad \frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{\mu_r}{\theta} = \lim_{x_r \to \bar{x}_r^-} \frac{\frac{1}{\varepsilon_r} + \frac{1}{\rho_r}}{1 - \frac{\theta}{\rho_r}} = \frac{\lim_{x_r \to \bar{x}_r^-} \sigma_r}{1 - 0} = \sigma_r(\bar{x}_r) > 0,$$

which immediately tells us that equation (11) in Ramsey (1927, p.56) or equation (C9) cannot produce a fruit of the proportionality at all on the sum of reciprocal price elasticities like $\mu_r \not\propto \frac{1}{\varepsilon_r} + \frac{1}{\rho_r}$ although equations (C15), (21) through (25) can produce it on the length elasticity of width θ in equation (C3) like $\mu_r \propto \theta$, on the tax revenue R_r like $\mu_r \propto R_r$ in equation (26), on the percentage-reduction K_r like $\mu_r \propto K_r$ in equation (A4), and so forth.

In terms of *Infinitesimal Calculus* as Keith D. Stroyan (1993, pp.37-42), by the way, it can be seen that a quotient limit from equation (11) in Ramsey (1927, p.56) or our equation (C9) diverges as

$$\text{(C16)} \quad \lim_{x_r \to \bar{x}_r^-} \frac{\frac{1}{\varepsilon_r} + \frac{1}{\rho_r}}{\mu_r} = \lim_{x_r \to \bar{x}_r^-} \frac{1 - \frac{\theta_r}{\rho_r}}{\theta_r} = \frac{1 + \lim\limits_{x_r \to \bar{x}_r^-} \frac{\lambda_r \phi_r'}{\lambda_r' \phi_r}}{\lim\limits_{x_r \to \bar{x}_r^-} \frac{-\lambda_r}{\lambda_r' x}} \equiv \frac{1 + \zeta_{\lambda_r}^+}{\zeta_{\theta_r}^+} \to +\infty$$

where we treat the length λ_r in equation (1) or (3) as an infinitesimal from the right denoted by $\zeta_{\lambda_r}^+$ such that $0 < \zeta_{\lambda_r}^+ \approx 0$ as well as its corresponding length elasticity θ_r in equation (C3) as an infinitesimal from the right of $\zeta_{\theta_r}^+$ such that $0 < \zeta_{\theta_r}^+ \approx 0$ due to equation (22) as $\lim\limits_{x_r \to \bar{x}_r^-} \lambda_r = \phi_r(\bar{x}_r) - f_r(\bar{x}_r) = 0$ and to the limit of equation (C3) as $\lim\limits_{x_r \to \bar{x}_r^-} \theta_r = \dfrac{-\{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)\}}{\{\phi_r'(\bar{x}_r) - f_r'(\bar{x}_r)\}\, \bar{x}_r} = 0$. As proved in proposition 1, therefore, the sum function $\sigma_r = \dfrac{1}{\varepsilon_r} + \dfrac{1}{\rho_r}$ in equation (28) is always too large as infinity in equation (C16) to have the same order as the limit of equation (C3) and equations (21) through (25).

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