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**Optimal Progressive Taxation on Incomes  
with or without  
Automatic Transfer Payments to the Poor**

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# Optimal Progressive Taxation on Incomes with or without Automatic Transfer Payments to the Poor

by Hiroaki Fujimoto and Yui Nakamura\*

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## Abstract

*✍ This paper considers an optimization problem where a government collects income tax as her gross revenue  $R$ , e. g., to transfer financial provisions from its  $R$  to those who live in poverty. To investigate such taxation, we usually resort to “the principle of equal marginal sacrifice . . . , being derived from the utilitarian objective of the maximization of the sum of individual utilities” as discussed in Anthony B. Atkinson (1973, p.91). However, this principle “would involve lopping off the tops of all incomes above the minimum income and leaving everybody, after taxation, with equal incomes” as mentioned by Arthur C. Pigou (1949, p.57): I. e., the tops of the highest incomes are taken as her net revenue  $\underline{R}$ ; middle incomes are automatically transferred to the lowest ones till an equal income after tax is gained; which violates an injunction of ‘do not alter the inequality of incomes by taxation’ suggested by Hugh Dalton (1967, p.63). Then, Atkinson (ibid., p.92) says the principle “is dismissed by most authors.” Instead, we would like to show here with some closed-form solutions that a new rule of equal marginal sacrifice can exempt us from the Pigou-Dalton’s modifications, being derived from the libertarian cost-benefit objective of the maximization of the sum of market surpluses. (JEL D61, D63, H21, J20, P43)*

When we investigate a question of how income tax should be progressive, the literature often gives us four principles concerning ‘sacrifice’ to its tax-payment such as ‘equal sacrifice,’ ‘proportional sacrifice,’ ‘equal marginal sacrifice,’ and a fourth one as an injunction of ‘do not alter the inequality of incomes by taxation’ added by Hugh Dalton (1967, p.63; a word of “inequality” is changed into that of “distribution,” 2003, p.91).

According to, for example, Francis Y. Edgeworth (1925, pp.101-22), Hugh Dalton (1967, pp.68-9), and Anthony B. Atkinson (1973, p.91), a passage of “the great-happiness of the greatest number” by Jeremy Bentham as well as that of “whatever sacrifices it (a government) requires from them (classes of individuals) should be made to be bear as nearly as possible with the same pressure upon all” by John S. Mill are mathematically put into the first two principles.<sup>1</sup> The third of ‘equal marginal sacrifice’ is mathematically the most rational among them by maximizing the sum of common utility  $u$  with respect to tax rates subject to a government’s net revenue  $\underline{R}$ , not a gross one of  $R$ .

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<sup>1</sup>See our appendixes C and D for the principle of ‘equal sacrifice’ and that of ‘proportional sacrifice,’ respectively.

To see it for a while, assume that there were  $n$  individuals with common utility function  $u_1$  of a post-tax or disposable income  $z_i \equiv y_i - T_i = y_i - t_i \times y_i = (1 - t_i)y_i$  or  $u_1 \equiv \ln z_i$  where  $y_i$  is a pre-tax income,  $t_i$  is a tax rate,  $T_i$  is tax-payment as  $T_i \equiv t_i \times y_i$ , and  $i$  is a running index as  $i = 1, 2, \dots, n$ . The government is then assumed to maximize a sum of  $u_1$  with respect to the tax rate  $t_i$  subject to the net revenue  $\underline{R}$ . Denote by  $\lambda_1$  a Lagrange-multiplier, and a Lagrangian function  $\mathcal{L}_1$  becomes

$$(E.1) \quad \mathcal{L}_1 \equiv \sum_{i=1}^n \ln z_i + \lambda_1(\underline{R} - \sum_{i=1}^n T_i) = \sum_{i=1}^n \ln\{(1 - t_i)y_i\} + \lambda_1(\underline{R} - \sum_{i=1}^n t_i y_i), \quad 1 - t_i > 0.$$

As proved in appendix E, we have  $n + 1$  equations of the first-order condition for equation (E.1) as  $-\frac{1}{1 - t_i} = \lambda_1 y_i$  for  $i = 1, 2, \dots, n$  and  $\underline{R} = \sum_{i=1}^n t_i y_i$ ; in fact, the former  $n$  equations supply us with the principle of ‘equal marginal sacrifice’ of utility  $u_1$  with respect to tax-payment  $T_i$  as  $\frac{d u_1}{d T_i} = \frac{d u_1}{d t_i} \frac{1}{d T_i / d t_i} = -\frac{1}{1 - t_i} \frac{1}{y_i} = \lambda_1 < 0$  for all  $i$  like the Hermann Heinrich Gossen’s Second Law.<sup>2</sup> Because its optimal multiplier  $\lambda_1^*$  can be solved as  $\lambda_1^* \equiv -\frac{n}{\bar{y} - \underline{R}} < 0$  in equation (E.7), an optimal disposable income  $z_i^*$  of the  $i$ -th individual becomes identical to each other for all  $i = 1, 2, \dots, n$  as  $z_i^* = y_i(1 - t_i^*) = -\frac{1}{\lambda_1^*} = \frac{1}{n}(\bar{y} - \underline{R}) > 0$  where  $t_i^*$  is an optimal tax rate on income  $y_i$  as

$$(E.10) \quad t_i^* \equiv \frac{T_i^*}{y_i} = 1 - \frac{z_i^*}{y_i} = 1 - \frac{1}{n y_i}(\bar{y} - \underline{R}) \begin{cases} \geq 0; t_i^* < 1 & \text{if } y_i \geq (\bar{y} - \underline{R})/n, \\ < 0 & \text{if } 0 < y_i < (\bar{y} - \underline{R})/n, \end{cases}$$

and  $\bar{y}$  is an aggregated income level of  $\bar{y} \equiv \sum_{i=1}^n y_i$ .<sup>3</sup>

Without losing any generality, we can line up  $n$  optimal tax-payment  $T_i^* \equiv t_i^* y_i$  in ascending order with three groups denoted by  $T_S^*$ ,  $T_M^*$ , and  $T_H^*$  as displayed below:

$$(\star.0) \quad \overbrace{T_1^* \leq T_2^* \leq \dots \leq T_s^*}^{\text{Group } T_S^*} < 0 < \overbrace{T_{s+1}^* \leq T_{s+2}^* \leq \dots \leq T_{s+m}^*}^{\text{Group } T_M^*} < \overbrace{T_{n-h+1}^* \leq T_{n-h+2}^* \leq \dots \leq T_{n-1}^* \leq T_n^*}^{\text{Group } T_H^*}.$$

It can be now easily seen as Arthur C. Pigou (1949, pp.57-8) points out that the principle of ‘equal marginal sacrifice’ by utilitarians “would involve lopping off the tops of all incomes above the minimum income and leaving everybody, after taxation, with equal incomes” of  $z_i^* = \frac{1}{n}(\bar{y} - \underline{R})$  for all  $i = 1, 2, \dots, n$ : Namely, “the logical procedure would be first to take for the government’s needs the tops of the highest incomes” explicitly as positive taxes  $T_H^* > 0$  to sum up into the net revenue of  $\underline{R} = \sum T_H^*$ ; “and then to continue taxing middle grade incomes” implicitly as another positive taxes  $T_M^* \geq 0$ ; simultaneously and automatically “giving bounties from the proceeds to the smallest incomes” as negative taxes  $T_S^* < 0$  such that  $-\sum T_S^* = \sum T_M^*$  “till a dead level of equality is attained” at  $z_i = z_i^*$  as an optimal post-tax income for all  $i$ .

<sup>2</sup>The Law is known as in an equilibrium, specially at a stationary point of the first-order condition, a consumer should allocate expenditures so that the ratio of marginal utility to price is equal across all commodities as well as to a Lagrange-multiplier.

<sup>3</sup>Even if Daniel Bernoulli’s utility  $u_1 \equiv \ln z_i$  is replaced by  $u_2 \equiv c - \frac{1}{z_i}$  of Dalton (1967, p.69), then we will have exactly the same result as  $z_i^* = \frac{1}{n}(\bar{y} - \underline{R})$  due to ‘equal marginal sacrifice’ of  $\frac{d u_2}{d T_i}$  for all  $i = 1, 2, \dots, n$ . See appendix E② for details.

To see it further, assume the  $i$ -th individual earned an annual income by  $i$  much as  $y_i = i$ , or  $y_1 = 1, y_2 = 2, \dots$ , and  $y_n = n$ , then we can have an aggregate income level  $\bar{y} = \sum_{i=1}^n y_i = \sum_{i=1}^n i = \frac{n(n+1)}{2}$ .<sup>4</sup> Owing to its identical optimal disposable income of  $z_i^* = \frac{1}{n}(\bar{y} - \underline{R})$  for all  $i$ , the  $i$ -th individual's optimal tax-payment  $T_i^* = y_i - z_i^*$  is computed as  $T_i^* = i - \frac{n+1}{2} + \frac{\underline{R}}{n}$ .<sup>5</sup> Assume moreover that we had, for instance,  $n = 10$  and  $\underline{R} = 5$ , and so that its aggregate income  $\bar{y}$  and identical disposable income  $z_i^*$  could be calculated as  $\bar{y} = 55$  and  $z_i^* = 5$ , respectively. Just like equation (★.0), ten optimal tax payments  $T_i^*$  are lined up into

$$(★.1) \quad \underbrace{T_1^* = -4 < T_2^* = -3 < \dots < T_4^* = -1 < 0}_{\text{Group } T_S^* \text{ with } -\sum T_S^* = 10} = \underbrace{T_5^* < T_6^* = 1 < T_7^* = 2 < \dots < T_9^* = 4}_{\text{Group } T_M^* \text{ with } \sum T_M^* = 10} < \underbrace{T_{10}^* = 5 = \underline{R}}_{\text{Group } T_H^*}$$

in which we can numerically confirm the above Pigouvian “logical procedure (*ibid.*, p.58):” That is, only a part of the highest incomes has been seemingly left for the net revenue  $\underline{R}$ ; middle incomes were automatically transferred to the poor, which equivalently implies there was a gross revenue  $R$  such that  $R = \sum T_M^* + \underline{R}$ .<sup>6</sup>

According to Atkinson (1973, p.92), therefore, the principle of ‘equal marginal sacrifice’ of utility  $u$  with respect to tax-payment  $T_i$ , which has another name of the minimum-sacrifice theory, “has come under a great deal of attack and is dismissed by most authors. Two main lines of criticism may be distinguished:

(a) that the minimum-sacrifice theory takes no account of the possible disincentive effect of taxation ( $z(n)$  may be influenced by the tax structure).<sup>7</sup>

(b) that the underlying utilitarian framework is inadequate” because an automatic transfer-payment  $\sum T_M^*$  is hidden outside of a net constraint  $\underline{R}$ , because  $z(n)$  is approximative to the optimal one of  $z_i^* = \frac{1}{n}(\bar{y} - \underline{R})$ , because only a possible incentive effect on labor supply but any on demand seems to be explicitly included, and because an individual's utility is neither observable nor comparative with each other in general.<sup>8</sup>

To stay away from them, the rest of our paper is organized as follows. First of all, section I presents our model of income taxation where a government is supposed to intervene  $n$  labor markets and raise her

<sup>4</sup>Denote by  $\tilde{y}$  an approximation to an aggregate income  $\bar{y}$  of the summation of the  $i$ -th individual's income  $y_i$  or  $\bar{y} \equiv \sum_{i=1}^n y_i$ , integrating a product of an income function  $y_r$  of a rank  $r$  and its frequency function  $f(r)$  as  $\tilde{y} \equiv \int_0^T y_r f(r) dr$  where  $y_r \equiv \frac{dy_r}{dr} \geq 0$  and  $\int_0^n f(r) dr = n$ . In this case of  $\bar{y} = \frac{n(n+1)}{2}$ , we can make it as  $\tilde{y} \approx \frac{n^2}{2} = \tilde{y} \equiv \int_0^n r dr = [\frac{r^2}{2}]_0^n$  where we have not merely  $y_r = 1 > 0$  with  $y_r = r$  but also  $\int_0^n 1 dr = [r]_0^n = n$  with  $f(r) = 1$ , reflecting only one  $r$ -th individual has the income of  $y_r = r$ .

<sup>5</sup>Because we are able to obtain an approximation  $z(n)$  on the optimal post-tax income  $z_i^*$  as  $z_i^* = \frac{1}{n}(\bar{y} - \underline{R}) \approx z(n) = \frac{1}{n}(\tilde{y} - \underline{R})$  where  $\tilde{y} = \frac{n^2}{2}$  derived from footnote 4, the optimal tax-payment  $T_i^*$  can be also approximated to  $T(n) \equiv i - \frac{n}{2} + \frac{\underline{R}}{n}$  at once.

<sup>6</sup>Other things being equal, assume an income function were  $y_i = i^2$  instead of  $y_i = i$ , and we can compute its aggregate level as  $\bar{y} = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ . Substituting  $n = 10$  and  $\underline{R} = 5$  into  $z_i^* = \frac{1}{n}(\bar{y} - \underline{R})$ , we have such an identical level of  $z_i^* = 38$  that it can be seen if  $i \geq 7$ , then  $i^2 > 38$ , and a gross one becomes  $R = 11 + 26 + 43 + 62 = 142$  with a hidden  $\sum T_M^* = 137$ . So, it is easy to see under the principle of ‘equal marginal sacrifice’ that  $z_i^*$  cannot be influenced by any tax structure at all.

<sup>7</sup>Recall in footnotes 4 and 5 that  $z(n) = \frac{n}{2} - \frac{\underline{R}}{n} = 4.5 < z_i^* = 5$  if  $y_i = i, n = 10$ , and  $\underline{R} = 5$ . On the contrary to that “ $z(n)$  may be influenced by the tax structure,” this approximative income  $z(n)$  is also a variable to be solved in terms of parameters initially given in the model so that  $z(n)$  cannot be influenced by another parameter, for example, such as  $\kappa_1, \kappa_2, \kappa_3$ , or  $\kappa_4$  in our appendixes C and D, which characterizes a tax structure under the principle of ‘equal sacrifice’ or ‘proportional sacrifice.’ Neither can be the optimal constant solution  $z_i^* = \frac{1}{n}(\bar{y} - \underline{R})$  as mentioned in footnote 6. It may be worth noticing here that some economists use a word of ‘solution’ as some parts of the first-order condition, which is often left unsolved in terms of parameters initially given in their models as put in Atkinson (1973, p.91) that the “solution is straightforward (on the assumption that  $U$  is concave):  $U_n'[z(n) - T(n)]$  equal all  $n$ ,” which approximately corresponds to equations (E.4) and (E.20) in our appendix E.

<sup>8</sup>For further discussions, see, for example, John Creedy (1986, p.117), Lionel Robbins (1949, pp.136-43), Amartya K. Sen (1970, p.106), Robert Cooter (1978, p.756), James A. Mirrlees (1971, pp.175-7), *etc.*, respectively.

tax revenue by a gross amount of  $R$ . Hence, our model belongs to the cost-benefit analysis. Next, section II studies a few of closed-form solutions for those  $n$  labor markets to an annual-salary or hourly-wage system based upon our new rule of the libertarian ‘equal marginal sacrifice’ of market-surplus  $ms_i$  with respect to tax-payment  $T_i$ . Finally, section III concludes this paper: That is, the utilitarian ‘equal marginal sacrifice’ of utility  $u$  with respect to that  $T_i$  “dismissed by most authors (Atkinson, 1973, p.92)” cannot exempt us from the Pigou-Dalton’s modifications; but our libertarian rule can keep us away from them, and so forth.

We put all proofs in appendix A. The other four appendixes are also available for a bordered Hessian in appendix B, for the principle of ‘equal sacrifice’ in appendix C, for the principle of ‘proportional sacrifice’ in appendix D, and for the principle of ‘equal marginal sacrifice’ in appendix E, respectively.

## I. The Model

### A. $N$ Labor Markets

Suppose there were  $n$  individuals with a running index  $i$ , namely,  $i = 1, 2, \dots, n$ , the  $i$ -th of who had her or his own labor supply function  $S_i$  as follows:

$$(1) \quad w_i = S_i(x_i) > 0; \quad S_i(0) = 0; \quad S'_i \equiv \frac{dS_i}{dx_i} > 0,$$

where  $x_i (> 0)$  is a labor quantity or an amount of labor of the  $i$ -th individual, which may take different unit from each other such as hour, week, month, year, *etc.*, and  $w_i$  is a wage rate per one unit of the quantity, respectively. Denote by  $D_i$  an inverse demand function of labor quantity  $x_i$  as follows:

$$(2) \quad w_i = D_i(x_i) > 0; \quad D'_i \equiv \frac{dD_i}{dx_i} \leq 0,$$

then the  $i$ -th individual’s initial equilibrium wage rate  $\bar{w}_i$  and quantity of labor  $\bar{x}_i$  are given by that supply is equal to demand, or equation (1) = equation (2) gives us that

$$(3) \quad \bar{w}_i \equiv S_i(\bar{x}_i) = D_i(\bar{x}_i).$$

So, we can have the following domain of labor quantity  $x_i$  as well as her or his pre-tax income level  $y_i$ :<sup>9</sup>

$$(4) \quad 0 < x_i \leq \bar{x}_i;$$

$$(5) \quad y_i \equiv \bar{w}_i \times \bar{x}_i.$$

### B. Income Taxation

Suppose that a government intervened the above  $n$  labor markets in order to impose a tax rate  $t_i$  on the  $i$ -th individual’s (pre-tax) income  $y_i$  and collect an amount of  $R$  as her revenue, in which the rate  $t_i$  is a choice variable to make its income level  $y_i$  shrunk to a disposable one or post-tax income, say  $z_i$  as follows:

$$(6) \quad z_i \equiv (1 - t_i) y_i = y_i - T_i; \quad 0 \leq t_i < 1,$$

<sup>9</sup>As the quantity  $x_i$  may take different unit, we can replace it by a unit-less quantity of labor  $q_i \equiv \frac{x_i}{\bar{x}_i}$  such that  $0 < q_i \leq 1$ .

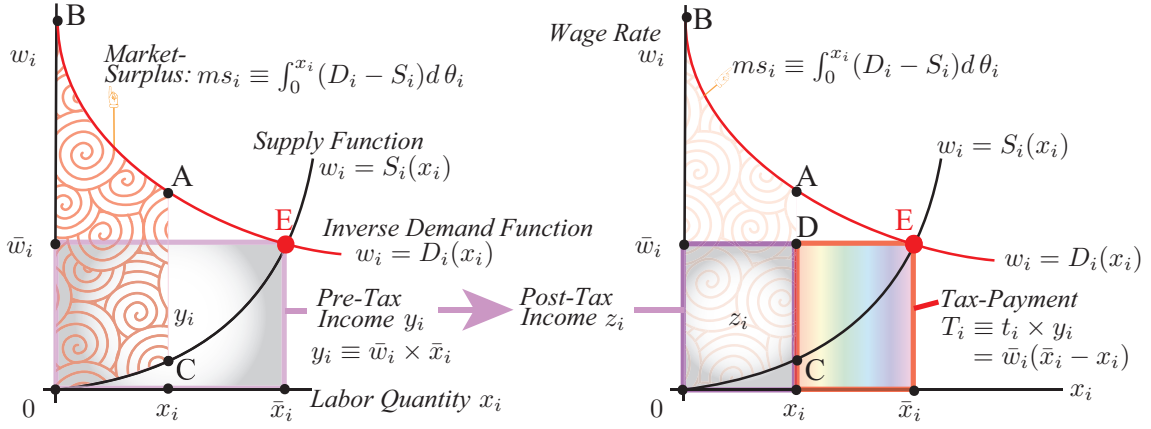


Figure 1: A Rectangle Area of Tax-Payment  $T_i$  with an Income Tax Rate  $t_i$

where  $T_i \equiv t_i \times y_i$  is tax-payment. As shown in figure 1 where Point **E** is the equilibrium point that yields equation (3), it is easy to contemplate that the tax-payment  $T_i$  can be expressed in terms of a labor quantity  $x_i$  within the domain of equation (4) as  $0 < x_i \leq \bar{x}_i$ , or from equation (5) of  $y_i = \bar{w}_i \bar{x}_i$  that

$$(7) \quad T_i \equiv t_i y_i = \bar{w}_i(\bar{x}_i - x_i) = y_i - \bar{w}_i x_i,$$

which intuitively reflects that at the last moment of the labor quantity  $\bar{x}_i - x_i$ , any tax-payer does not work for herself nor himself but works for the government as a patron. Since equations (7), (5), and (4) give us

$$(8) \quad t_i \equiv \frac{T_i}{y_i} = \frac{y_i - \bar{w}_i x_i}{y_i} = 1 - \frac{x_i}{\bar{x}_i}, \quad 0 < x_i \leq \bar{x}_i,$$

by taking its derivative with respect to  $x_i$ , we can have the following fact of strict monotone decreasing as

$$(9) \quad \frac{dt_i}{dx_i} = -\frac{1}{\bar{x}_i} < 0; \quad 0 < x_i \leq \bar{x}_i,$$

which tells us that the labor quantity  $x_i$  can be employed for a choice variable instead of the rate  $t_i$ .<sup>10</sup> So, like an area **A•B•C** as drawn in figure 1, let  $ms_i$  be the  $i$ -th individual's market-surplus of

$$(10) \quad ms_i \equiv \int_0^{x_i} \{D_i(\theta_i) - S_i(\theta_i)\} d\theta_i,$$

and let  $g$  be a sum of tax payments  $T_i$  in equation (7) or

$$(11) \quad g \equiv \sum_{i=1}^n T_i = \sum_{i=1}^n (y_i - \bar{w}_i x_i) = \bar{y} - \sum_{i=1}^n \bar{w}_i x_i,$$

where  $\bar{y}$  is an aggregated income level of  $\bar{y} \equiv \sum_{i=1}^n y_i$ , then the government is now able to take all  $x_i$ 's as her choice variables so as to maximize a sum of equation (10) as her objective function subject to a constant tax revenue, say  $R > 0$ : That is, she

$$(12) \quad \text{maximizes } \mathcal{L} \equiv \sum_{i=1}^n \int_0^{x_i} \{D_i(\theta_i) - S_i(\theta_i)\} d\theta_i + \ell (R - \bar{y} + \sum_{i=1}^n \bar{w}_i x_i),$$

<sup>10</sup>As noted in footnote 9, we can replace  $x_i$  by  $q_i \equiv \frac{x_i}{\bar{x}_i}$ , or  $x_i = \bar{x}_i q_i$  such that  $0 < q_i \leq 1$ . The unit-less quantity  $q_i$  can be also used for a choice variable due to the chain-rule of  $\frac{dt_i}{dq_i} = \frac{dt_i}{dx_i} \frac{dx_i}{dq_i} = -1 < 0$  or to equation (8) as  $t_i = 1 - q_i$ .

where  $\mathcal{L}$  is a Lagrange-function and  $\ell$  is a Lagrange-multiplier, respectively.<sup>11</sup>

### C. The First-Order Condition

By taking partial derivatives of equation (12) with respect to not only  $n$  choice variables,  $x_1, x_2, \dots, x_n$  but also the multiplier  $\ell$ , we can obtain a set of  $n + 1$  equations for the first-order condition as follows:

$$(13) \quad \frac{\partial \mathcal{L}}{\partial x_i} = D_i(x_i) - S_i(x_i) + \ell \bar{w}_i = 0 \text{ for } i = 1, 2, \dots, n;$$

$$(14) \quad \frac{\partial \mathcal{L}}{\partial \ell} = R - \bar{y} + \sum_{i=1}^n \bar{w}_i x_i = 0.$$

Then, equation (13) provides us with a new rule about equal marginal sacrifice: *I. e.*,

**PROPOSITION 1:** *In a post-tax equilibrium, the following  $i$ -th individual's marginal market-surplus  $ms_i$  with respect to her or his tax-payment  $T_i$  to the government, or*

$$(15) \quad \frac{dms_i}{dT_i} = \frac{dms_i}{dx_i} \frac{1}{dT_i/dx_i} = \ell \leq 0$$

*equals each other to the Lagrange-multiplier  $\ell$  for all  $i = 1, 2, \dots, n$  owing to the derivatives  $\frac{dms_i}{dx_i} = D_i(x_i) - S_i(x_i) \geq 0$  and  $\frac{dT_i}{dx_i} = -\bar{w}_i < 0$  from equations (10) and (7), respectively.*

*Proof.* See appendix A in details. □

So, it can be observed in proposition 1 that the government should impose an amount of tax  $T_i$  on the  $i$ -th individual's income  $y_i$  so that a ratio of marginal market-surplus  $ms_i$  to the pre-tax wage rate  $\bar{w}_i$  is equal across all labor quantities  $x_i$  as well as to the multiplier  $\ell$ , which reflects that the  $i$ -th individual's sacrifice or negative market-surplus ( $-ms_i$ ) equally goes up by  $\ell$  units when the tax-payment  $T_i$  goes up by one unit.

### D. The Second-Order Condition

By taking partial derivatives of equations (11) and (13) with respect to  $x_i$ , we can acquire  $2n$  non-zero elements of a bordered Hessian, say  $|\bar{H}|$  for the second-order condition as follows: For all  $i = 1, 2, \dots, n$ ,

$$(16) \quad g_i \equiv \frac{\partial g}{\partial x_i} = \frac{d}{dx_i} g = \frac{d}{dx_i} (\bar{y} - \sum_{i=1}^n \bar{w}_i x_i) = -\bar{w}_i < 0;$$

$$(17) \quad \mathcal{L}_{ii} \equiv \frac{\partial^2 \mathcal{L}}{\partial x_i^2} = \frac{d}{dx_i} \left( \frac{\partial \mathcal{L}}{\partial x_i} \right) = \frac{d}{dx_i} \{D_i(x_i) - S_i(x_i) + \ell \bar{w}_i\} = D'_i(x_i) - S'_i(x_i) < 0$$

from equations (1) and (2):  $S' > 0$ ;  $D' \leq 0$ , respectively.

Because all we have to do is checking out whether or not all non-zero elements of the partial derivatives  $g_i$  and  $\mathcal{L}_{ii}$  are negative according to our appendix B with a mathematical induction, equations (16) and (17) are always satisfied with the second-order condition for a maximization problem of equation (12).

<sup>11</sup>Our model belongs to the cost-benefit analysis of  $n$  labor markets, each of which the government intervenes and collects its tax-payment  $T_i$  to sum up into her gross tax revenue  $R$ : *I. e.*, an employer's willingness to pay for the  $i$ -th individual's labor  $x_i$  is defined as benefits measured by an area of  $\bullet A \bullet B \bullet 0 \bullet x_i$  beneath  $D_i$  in figure 1; whereas this individual's willingness to accept as an employee is as costs by that of  $\bullet C \bullet 0 \bullet x_i$  beneath  $S_i$ , which yields that of  $\bullet A \bullet B \bullet 0 \bullet C$  as market-surplus in equation (10) reaching its maximum at  $x_i = \bar{x}_i$  where demand equals supply by that  $\frac{dms_i}{dx_i} = D_i(x_i) - S_i(x_i) = 0$  with  $\frac{d^2ms_i}{dx_i^2} = D'_i - S'_i < 0$ .

## II. Some Closed-Form Solutions

### A. *N* Affine, Linear Plus Intercepts, Labor Markets

Suppose the  $i$ -th individual's labor market for  $i = 1, 2, \dots, n$  had not only a supply function:

$$(18) \quad S_i(x_i) \equiv a_i x_i$$

with parameters  $a_i > 0$  for slopes such that  $S'_i \equiv \frac{dS_i}{dx_i} = a_i > 0$  but also an affine inverse demand function:

$$(19) \quad D_i(x_i) \equiv -b_i x_i + w_{0i}$$

with parameters  $b_i \geq 0$  for slopes and  $w_{0i} > 0$  for intercepts such that  $D'_i \equiv \frac{dD_i}{dx_i} = -b_i \leq 0$ . Every pre-tax equilibrium quantity of labor  $\bar{x}_i^A$  can be calculated from equations (18) and (19) of  $a_i \bar{x}_i^A = -b_i \bar{x}_i^A + w_{0i}$  as

$$(20) \quad \bar{x}_i^A \equiv \frac{1}{a_i + b_i} w_{0i},$$

which yields its wage rate  $\bar{w}_i^A$  to us with equations (20) and (18) or (19) of  $\bar{w}_i^A = a_i \bar{x}_i^A = -b_i \bar{x}_i^A + w_{0i}$  that

$$(21) \quad \bar{w}_i^A \equiv \frac{a_i}{a_i + b_i} w_{0i} = c_i^A w_{0i}, \quad 0 < c_i^A \leq 1,$$

where  $c_i^A$  is a coefficient defined as  $c_i^A \equiv \frac{a_i}{a_i + b_i}$  ( $= \frac{\bar{w}_i^A}{w_{0i}}$ ), which has a region of  $0 < c_i^A \leq 1$  since we have already assumed for slopes that  $0 < a_i \leq a_i + b_i$  and  $0 \leq b_i$ .

Both a domain of a post-tax equilibrium quantity of labor  $x_i$  and a pre-tax income level  $y_i^A$  of the  $i$ -th individual can be easily calculated from equations (4) and (5) as follows:

$$(22) \quad 0 < x_i \leq \bar{x}_i^A = \frac{1}{a_i + b_i} w_{0i};$$

$$(23) \quad y_i^A \equiv \bar{w}_i^A \times \bar{x}_i^A = \frac{a_i}{(a_i + b_i)^2} w_{0i}^2 \quad (\leq \frac{1}{4} \frac{w_{0i}^2}{b_i} \text{ as its maximum if } b_i = a_i > 0).$$

So, it can be also easily shown from equation (15) in proposition 1 that

$$(24) \quad \frac{-b_i x_i + w_{0i} - a_i x_i}{-\bar{w}_i^A} = \ell,$$

which becomes with equations (20), (21), and (23) that

$$(25) \quad x_i = c_i^A \bar{x}_i^A \ell + \bar{x}_i^A = \frac{\alpha_i^A}{\bar{w}_i^A} \ell + \bar{x}_i^A,$$

where  $\alpha_i^A$  is the  $i$ -th individual's 'ability' of a maximum amount potentially payable to a government in each labor market defined as follows:<sup>12</sup>

$$(26) \quad \alpha_i^A \equiv c_i^A y_i^A \leq y_i^A,$$

then we have the following corollary for the  $n$  affine labor markets consist of equations (18) and (19).

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<sup>12</sup>A word of 'ability' is stemmed from one of four maxims put by Adam Smith (1958, p.307) as the  $n$  individuals or "subjects of every state ought to contribute towards the support of the government, as nearly as possible, in proportion to their respective abilities; that is, in proportion to the revenue which they respectively enjoy under the protection of the state." So, let  $\pi$  be a proportion rate. We may have it as  $\hat{\pi} = \frac{R}{\sum_{i=1}^n y_i}$  due to a revenue  $R = \sum_{i=1}^n T_i$  with a tax-payment  $T_i = \pi y_i$  and an income  $y_i$ .



**COROLLARY 1:** In this case  $A$  of  $n$  labor markets, a government should impose an optimal tax-payment  $T_i^{A*}$ , which sums up into a certain gross tax revenue  $R$ , on the  $i$ -th individual's pre-tax income  $y_i^A$  as

$$(27) \quad T_i^{A*} \equiv \frac{\alpha_i^A}{\sum_{i=1}^n \alpha_i^A} R = \frac{c_i^A y_i^A}{\sum_{i=1}^n c_i^A y_i^A} R > 0$$

for all  $i = 1, 2, \dots, n$ .

*Proof.* See appendix A in details. □

In other words, the  $i$ -th individual should pay the optimal income tax  $T_i^{A*}$  for the amount of  $R$  at a weighted 'ability' portion  $\frac{\alpha_i^A}{\sum_{i=1}^n \alpha_i^A}$  or at a ratio of the 'ability'  $\alpha_i^A$  of the  $i$ -th individual in equation (26) to that of all  $n$  individuals  $\sum_{i=1}^n \alpha_i^A$ , naming it a weighted 'ability' rule after that portion. Accordingly, the government is now able to obtain her gross revenue  $R$  from among  $n$  individuals in descending order of the biggest 'ability'  $\alpha_i^A$  ( $= c_i^A y_i^A \leq y_i^A \leq \frac{1}{4b_i} w_{0i}^2$ ;  $b_i = a_i > 0$ ) potentially payable to her as the  $i$ -th individual's income tax.<sup>13</sup>

Substituting equation (27) into equations (6) and (8), respectively, we can compute an optimal post-tax or disposable income  $z_i^{A*}$  of the  $i$ -th individual and an optimal tax rate  $t_i^{A*}$  as follows:<sup>14</sup>

$$(28) \quad z_i^{A*} \equiv y_i^A - T_i^{A*} = y_i^A - \frac{\alpha_i^A}{\sum_{i=1}^n \alpha_i^A} R > 0;$$

$$(29) \quad t_i^{A*} \equiv \frac{T_i^{A*}}{y_i^A} = \frac{c_i^A}{\sum_{i=1}^n c_i^A} R > 0,$$

the latter of which can be treated as a constant because of a sum of all individuals' 'abilities' or  $\sum_{i=1}^n \alpha_i^A$ .<sup>15</sup> Consequently, some cross-sectional studies to equations (27) through (29), in which the sum of  $\sum_{i=1}^n \alpha_i^A$  is assumed to be constant, can tell us that

$$(30) \quad \frac{dT_i^{A*}}{dy_i^A} = \frac{c_i^A}{\sum_{i=1}^n c_i^A} R = t_i^{A*} > 0 \text{ and } \frac{d^2 T_i^{A*}}{d(y_i^A)^2} = 0;$$

$$(31) \quad \frac{dz_i^{A*}}{dy_i^A} = 1 - t_i^{A*} > 0 \text{ and } \frac{d^2 z_i^{A*}}{d(y_i^A)^2} = 0;$$

$$(32) \quad \frac{dt_i^{A*}}{dy_i^A} = 0 \text{ and } \frac{d^2 t_i^{A*}}{d(y_i^A)^2} = 0;$$

<sup>13</sup>Using another running index of  $r = 1, 2, \dots, n$ , recall each definition of the coefficient  $c_r^A$ , the pre-tax income  $y_r^A$ , and the 'ability'  $\alpha_r^A$  of the  $r$ -th individual in equations (21), (23), and (26). Provided for arbitrary elements  $i, j$ , and  $k \in$  a set  $\{r | r = 1, 2, \dots, n\}$  that  $1 \leq \frac{y_j^A}{y_i^A} < \frac{c_i^A}{c_j^A}$  as well as  $1 \leq \frac{y_j^A}{y_k^A} \leq \frac{c_j^A y_j^A}{c_k^A y_k^A}$ , we can show their orders like that  $y_k^A, y_i^A \leq y_j^A$ , that  $c_k^A \leq c_j^A < c_i^A \leq 1$ , and so that  $c_k^A y_k^A \leq c_j^A y_j^A < c_i^A y_i^A$  or  $\alpha_k^A \leq \alpha_j^A < \alpha_i^A$ , which implies we can treat those 'abilities' as already reordered.

<sup>14</sup>It is interesting to see in equations (27) through (29) that if the coefficient  $c_i^A = \frac{\alpha_i^A}{a_i + b_i}$  in equation (21) became a constant, for instance, such as  $c_i^A = 1$  with  $b_i = 0$ ,  $c_i^A = \frac{1}{2}$  with  $b_i = a_i > 0$ , and so on for all  $i = 1, 2, \dots, n$ , then not only a weighted portion of  $\frac{\alpha_i^A}{\sum_{i=1}^n \alpha_i^A}$  could be reduced into that of  $\frac{y_i^A}{\sum_{i=1}^n y_i^A}$  but also another one of  $\frac{c_i^A}{\sum_{i=1}^n c_i^A}$  could be done into that of  $\frac{1}{\sum_{i=1}^n y_i^A}$ . So, a constant tax rate in equation (29), say  $t_i^{A**} \equiv \frac{R}{\sum_{i=1}^n y_i^A}$  here can be a solution to the maxim of Smith in footnote 12.

<sup>15</sup>Recall in the utilitarian case or equation (E.8) of our appendix E that the post-tax income  $z_i^* \equiv \frac{1}{n}(\bar{y} - R)$  becomes a constant because all pre-tax incomes  $y_i$  are initially given to be fixed at the same point in time so that their sum or aggregate level  $\bar{y} \equiv \sum_{i=1}^n y_i$  should be also fixed as a constant in the cross-sectional study unlike in the ordinary comparative-static analysis.

however, some comparative-static analyses to them can tell us with another index  $r$  used in footnote 13 that for all  $r = 1, 2, \dots, n$ ,

$$(33) \quad \frac{dT_r^{A*}}{dy_r^A} = \frac{c_r^A \sum_{j \neq r} c_j^A y_j^A}{(\sum_{r=1}^n \alpha_r^A)^2} R > 0 \text{ and } \frac{d^2 T_r^{A*}}{d(y_r^A)^2} = -\frac{2(c_r^A)^2 \sum_{j \neq r} c_j^A y_j^A}{(\sum_{r=1}^n \alpha_r^A)^3} R < 0;$$

$$(34) \quad \frac{dz_r^{A*}}{dy_r^A} = 1 - t_r^{A*} + \frac{(c_r^A)^2 y_r^A}{(\sum_{r=1}^n \alpha_r^A)^2} R > 0 \text{ and } \frac{d^2 z_r^{A*}}{d(y_r^A)^2} = \frac{2(c_r^A)^2 \sum_{j \neq r} c_j^A y_j^A}{(\sum_{r=1}^n \alpha_r^A)^3} R > 0;$$

$$(35) \quad \frac{dt_r^{A*}}{dy_r^A} = -\frac{(c_r^A)^2}{(\sum_{r=1}^n \alpha_r^A)^2} R < 0 \text{ and } \frac{d^2 t_r^{A*}}{d(y_r^A)^2} = \frac{2(c_r^A)^3}{(\sum_{r=1}^n \alpha_r^A)^3} R > 0.$$

So, we can see in equation (28) unlike in equation (E.8) that the libertarian ‘equal marginal sacrifice’ would not “involve lopping off the tops of all incomes above the minimum income and leaving everybody, after taxation, with equal incomes (Pigou, 1949, p.57),” besides, in equation (31) unlike equation (E.11) that this libertarian rule would not “alter the inequality of incomes by taxation (Dalton, 1967, p.63),” either, moreover, in equation (34) unlike (E.14) that under such a libertarian government, everybody would progressively have an incentive to work harder than before, *etc.*

### B. *N Labor Markets with Hyperbolic Labor Demands*

In this section B, let us replace only the inverse demand function of

$$(19) \quad D_i(x_i) \equiv -b_i x_i + w_{0i}$$

used in the previous section A by the following hyperbolic one of

$$(36) \quad D_i(x_i) \equiv \frac{h_i}{x_i}$$

with a parameter  $h_i > 0$  such that  $D'_i \equiv \frac{dD_i}{dx_i} = -\frac{h_i}{x_i^2} < 0$  in order to study an annual-salary system where the  $i$ -th individual’s annual-salary  $h_i$  would be fixed no matter how many hours  $x_i$  she or he might supply an employer with them as her or his labor quantity since the  $i$ -th individual’s wage rate  $w_i$  is flexibly determined as  $w_i = D_i = S_i$  in equation (3) for all  $i = 1, 2, \dots, n$ , she or he always has the fixed amount of  $h_i = w_i \times x_i$  a year. Because we employ exactly the same supply function of equation (18) for here as

$$(18) \quad S_i(x_i) \equiv a_i x_i$$

with a parameter  $a_i > 0$  for a slope such that  $S'_i \equiv \frac{dS_i}{dx_i} = a_i > 0$ , each pre-tax equilibrium quantity of labor  $\bar{x}_i^B$  can be calculated from equations (18) and (36) of  $a_i \bar{x}_i^B = \frac{h_i}{\bar{x}_i^B}$  as

$$(37) \quad \bar{x}_i^B \equiv \sqrt{h_i/a_i},$$

which gives us the following wage rate  $\bar{w}_i^B$  with equations (37) and (18) or (36) of  $\bar{w}_i^B = a_i \bar{x}_i^B = \frac{h_i}{\bar{x}_i^B}$  as

$$(38) \quad \bar{w}_i^B \equiv \sqrt{a_i h_i}.$$

From equations (4) and (5), we calculate a domain of a post-tax equilibrium quantity of labor  $x_i$  and a pre-tax income level  $y_i^B$  of the  $i$ -th individual as follows:

$$(39) \quad 0 < x_i \leq \bar{x}_i^B = \sqrt{h_i/a_i};$$

$$(40) \quad y_i^B \equiv \bar{w}_i^B \times \bar{x}_i^B = h_i.$$

From equation (15) in proposition 1, then we also have that

$$(41) \quad \frac{h_i/x_i - a_i x_i}{-\bar{w}_i^B} = \ell,$$

which becomes a quadratic equation as  $a_i x_i^2 - \bar{w}_i^B \ell x_i - h_i = 0$  with a positive solution of

$$(42) \quad x_i = \frac{\ell + \sqrt{\ell^2 + 4}}{2 a_i} \bar{w}_i^B = \frac{\ell + \sqrt{\ell^2 + 4}}{2} \bar{x}_i^B = \frac{\ell + \sqrt{\ell^2 + 4}}{2} \frac{\alpha_i^B}{\bar{w}_i^B},$$

where  $\alpha_i^B$  is the  $i$ -th individual's 'ability' of a maximum amount potentially payable to a government as

$$(43) \quad \alpha_i^B \equiv y_i^B, \text{ her or his pre-tax income itself.}$$

Now, we can acquire a corollary for the  $n$  labor markets consist of equations (18) and (36) as follows:

**COROLLARY 2:** *In this case B of those markets with hyperbolic labor demand functions, a government should impose an optimal tax-payment  $T_i^{B*}$ , which sums up into a certain gross tax revenue  $R$ , on the  $i$ -th individual's pre-tax income  $y_i^B$  as*

$$(44) \quad T_i^{B*} \equiv \frac{\alpha_i^B}{\sum_{i=1}^n \alpha_i^B} R = \frac{y_i^B}{\sum_{i=1}^n y_i^B} R > 0$$

for all  $i = 1, 2, \dots, n$ .

*Proof.* See appendix A in details. □

This time, the  $i$ -th individual should pay the optimal income tax  $T_i^{B*}$  for the amount of  $R$  at a weighted 'ability' portion  $\frac{\alpha_i^B}{\sum_{i=1}^n \alpha_i^B}$  or at a ratio of an income  $y_i^B$  of the  $i$ -th individual in equation (40) to an aggregate income level of all  $n$  individuals  $\sum_{i=1}^n y_i^B$ . Therefore, the government is here able to simply collect her gross revenue  $R$  in descending order of the biggest income  $y_i^B$  from among  $n$  individuals.

By plugging equation (44) into equations (6) and (8), next, we can have an optimal post-tax or disposable income  $z_i^{B*}$  of the  $i$ -th individual and an optimal tax rate  $t_i^{B*}$  as follows:

$$(45) \quad z_i^{B*} \equiv y_i^B - T_i^{B*} = y_i^B - \frac{y_i^B}{\sum_{i=1}^n y_i^B} R > 0;$$

$$(46) \quad t_i^{B*} \equiv \frac{T_i^{B*}}{y_i^B} = \frac{1}{\sum_{i=1}^n y_i^B} R > 0,$$

the latter of which is exactly the same constant rate as Adam Smith's  $\hat{\pi}$  and some affine markets'  $t_i^{A**}$  as shown in footnotes 12 and 14, respectively.

Let  $\bar{y}$  be a sum of all individuals' 'abilities' or incomes as  $\bar{y} \equiv \sum_{i=1}^n \alpha_i^B = \sum_{i=1}^n y_i^B$ , then some cross-sectional studies to equations (44) through (46), in which its aggregate  $\bar{y}$  is supposed to be constant, can tell us that

$$(47) \quad \frac{dT_i^{B*}}{dy_i^B} = \frac{1}{\bar{y}} R = t_i^{B*} > 0 \text{ and } \frac{d^2 T_i^{B*}}{d(y_i^B)^2} = 0;$$

$$(48) \quad \frac{dz_i^{B*}}{dy_i^B} = 1 - t_i^{B*} > 0 \text{ and } \frac{d^2 z_i^{B*}}{d(y_i^B)^2} = 0;$$

$$(49) \quad \frac{dt_i^{B*}}{dy_i^B} = 0 \text{ and } \frac{d^2 t_i^{B*}}{d(y_i^B)^2} = 0;$$

whereas some comparative-static analyses to them can tell us with the index  $r$  used in footnote 13 that

$$(50) \quad \frac{dT_r^{B*}}{dy_r^B} = \frac{\sum_{j \neq r} y_j^B}{\bar{y}^2} R > 0 \text{ and } \frac{d^2 T_r^{B*}}{d(y_r^B)^2} = -\frac{2 \sum_{j \neq r} y_j^B}{\bar{y}^3} R < 0;$$

$$(51) \quad \frac{dz_r^{B*}}{dy_r^B} = 1 - t_r^{B*} + \frac{y_r^B}{\bar{y}^2} R > 0 \text{ and } \frac{d^2 z_r^{B*}}{d(y_r^B)^2} = \frac{2 \sum_{j \neq r} y_j^B}{\bar{y}^3} R > 0;$$

$$(52) \quad \frac{dt_r^{B*}}{dy_r^B} = -\frac{1}{\bar{y}^2} R < 0 \text{ and } \frac{d^2 t_r^{B*}}{d(y_r^B)^2} = \frac{2}{\bar{y}^3} R > 0$$

for all  $r = 1, 2, \dots, n$ . So, we are able to see in equations (45), (48), and (51) as seen in equations (28), (31), and (34) that the libertarian 'equal marginal sacrifice' in equation (15) of our proposition 1 would not "involve lopping off the tops of all incomes above the minimum income and leaving everybody, after taxation, with equal incomes (Pigou, 1949, p.57)," that the libertarian proposition would not "alter the inequality of incomes by taxation (Dalton, 1967, p.63)," and so that under such a libertarian government, everyone would progressively have an incentive to work harder than before.

### *C. N Labor Markets with Horizontal Labor Demands*

In this section C, rather than equations (19) and (36), we use the following inverse demand function of

$$(53) \quad D_i(x_i) \equiv w_{0i}$$

with a parameter  $w_{0i} > 0$  such that  $D'_i \equiv \frac{dD_i}{dx_i} = 0$  so as to study a kind of hourly-wage system where the  $i$ -th individual's wage rate  $w_{0i}$  per one unit of labor quantity  $x_i$  is fixed all the time. In addition to it, rather than equation (18), we use the following non-linear supply function of

$$(54) \quad S_i(x_i) \equiv a_i x_i^p$$

with parameters  $a_i > 0$  for slopes and  $p > 0$  for a common finite power such that  $S'_i \equiv \frac{dS_i}{dx_i} = p a_i x_i^{p-1} > 0$ . So that, we can calculate not only a pre-tax equilibrium quantity of labor  $\bar{x}_i^C$  by equalizing equations (53) to (54) or  $w_{0i} = a_i x_i^p$  as

$$(55) \quad \bar{x}_i^C \equiv (w_{0i}/a_i)^{1/p},$$

but also its wage rate  $\bar{w}_i^C$  by equations (53) and (54) again as  $\bar{w}_i^C = w_{0i} = a_i x_i^p$  or

$$(56) \quad \bar{w}_i^C \equiv w_{0i}.$$

From equations (4) and (5), we have a domain of a post-tax equilibrium quantity of labor  $x_i$  and a pre-tax income level  $y_i^C$  of the  $i$ -th individual as follows:

$$(57) \quad 0 < x_i \leq \bar{x}_i^C = (w_{0i}/a_i)^{1/p};$$

$$(58) \quad y_i^C \equiv \bar{w}_i^C \times \bar{x}_i^C = w_{0i}(w_{0i}/a_i)^{1/p}.$$

From equation (15) in proposition 1, then we also have that

$$(59) \quad \frac{w_{0i} - a_i x_i^p}{-\bar{w}_i^C} = \ell,$$

which becomes a polynomial of  $a_i x_i^p = \bar{w}_i^C (\ell + 1)$ , and it gives us the following

$$(60) \quad x_i = \bar{x}_i^C (\ell + 1)^{1/p} = \frac{\alpha_i^C}{\bar{w}_i^C} (\ell + 1)^{1/p},$$

where  $\alpha_i^C$  is the  $i$ -th individual's 'ability' of a maximum amount potentially payable to a government as

$$(61) \quad \alpha_i^C \equiv y_i^C, \text{ her or his pre-tax income itself.}$$

We are now able to obtain a corollary for the  $n$  labor markets consist of equations (53) and (54) of

**COROLLARY 3:** *In this case C of those markets with horizontal labor demand functions, a government should impose an optimal tax-payment  $T_i^{C*}$ , which sums up into a certain gross tax revenue  $R$ , on the  $i$ -th individual's pre-tax income  $y_i^C$  as*

$$(62) \quad T_i^{C*} \equiv \frac{\alpha_i^C}{\sum_{i=1}^n \alpha_i^C} R = \frac{y_i^C}{\sum_{i=1}^n y_i^C} R = \frac{y_i^C}{\bar{y}} R > 0$$

for all  $i = 1, 2, \dots, n$  where  $\bar{y}$  is an aggregate level of all  $n$  individuals' incomes  $y_i^C$  as  $\bar{y} \equiv \sum_{i=1}^n y_i^C$ .

*Proof.* See appendix A in details. □

Substituting equation (62) into equations (6) and (8), we are also able to obtain an optimal post-tax income  $z_i^{C*}$  of the  $i$ -th individual and an optimal tax rate  $t_i^{C*}$  as follows:

$$(63) \quad z_i^{C*} \equiv y_i^C - T_i^{C*} = y_i^C - \frac{y_i^C}{\sum_{i=1}^n y_i^C} R > 0;$$

$$(64) \quad t_i^{C*} \equiv \frac{T_i^{C*}}{y_i^C} = \frac{1}{\sum_{i=1}^n y_i^C} R > 0.$$

It is interesting to see in equations (62) through (64) that we have exactly the same solutions as observed in equations (44) through (46), the last of which is the same as Adam Smith's  $\hat{\pi}$  and some affine markets'  $t_i^{A**}$

as shown in footnotes 12 and 14, respectively. In this subsection, therefore, replacing the capital letter  $B$  of equations (47) through (52) by  $C$ , we can derive exactly the identical results from among the cross-sectional studies and comparative-static analyses.

### III. Concluding Remarks

In this paper, we have investigated an optimization problem of how income tax should be progressive,<sup>16</sup> in which a government is supposed to intervene  $n$  individuals' labor markets and impose taxes  $T_i$  on their incomes  $y_i$  by an amount of  $R = \sum_{i=1}^n T_i$  for the government, *e.g.*, to transfer financial provisions from its  $R$  to the poor.<sup>17</sup> In order to work out its optimization problem, in other words, we have used the libertarian cost-benefit objective as described in equations (10) through (12) and footnote 11 rather than the utilitarian one as shown in our appendix E.<sup>18</sup> There are several reasons why we have done so as follows:

1. That the utilitarian objective cannot include a gross tax revenue  $R$  but  $\underline{R}$ , which “denotes the net revenue to be raised” by the government as put in Anthony B. Atkinson (1973, p.91).
2. That “the underlying utilitarian framework is inadequate” as also put in Atkinson (*ibid.*, p.92) because automatic transfer payments  $\sum T_M^*$  are hidden outside of the net revenue  $\underline{R}$  as shown in equations (★.0) and (★.1) of our introductory part, because an approximative post-tax or disposable income, say  $z(n)$  done by utilitarians such as Francis Y. Edgeworth and James A. Mirrlees seems to involve only a possible incentive effect on labor supply but any on demand as mentioned in John Creedy (1986, pp.112-7),<sup>19</sup> and because an individual's utility is in general neither observable nor comparative with each other as stated in Lionel Robbins (1949, pp.136-43), *etc.*
3. That the utilitarian objective “would involve lopping off the tops of all incomes above the minimum income and leaving everybody, after taxation, with equal incomes” as put in Arthur C. Pigou (1949, p.57).
4. That economists and policymakers should “not alter the inequality of incomes by taxation” suggested by Hugh Dalton (1967, p.63; a word of “inequality” is changed into that of “distribution,” 2003, p.91).

To stay away from the above reasons or the principle of utilitarian ‘equal marginal sacrifice’ of utility  $u$  with respect to tax-payment  $T_i$ , first, our proposition 1 has provided us with a new rule of libertarian ‘equal marginal sacrifice’ of market-surplus  $ms_i$  with respect to that  $T_i$ , being derived from equation (13) as some parts of the first-order condition. As shown in our appendix B, second, the mathematical induction has told us that the libertarian objective of the maximization of the sum of market surpluses, or equation (12) is

<sup>16</sup>For indirect taxation, see, for example, Frank P. Ramsey (1927) as well as Hiroaki Fujimoto and Masahito Irie (2010).

<sup>17</sup>For poverty alleviation, income redistribution, and asymmetric information, see, *e.g.*, Yui Nakamura (2007) and (2009). By the way, even though we have treated  $R$  as a gross tax revenue for a government to transfer financial provisions from its  $R$  to those who live in poverty, how about including those who work on salaries for the government in  $R$ ? Namely, one may be interested in a question of what ‘ability’ is different between the poor and government employees, both of who do not seem to earn any value added eventually. However, we leave it open to question for future researches.

<sup>18</sup>For a further discussion about utilitarians versus libertarians, see, for example, Michael J. Sandel (2007) and (2010).

<sup>19</sup>Based upon our libertarian results, one may be interested in a question like what approximative post-tax income we should have instead of the one  $z(n)$  done by those utilitarians. But, we leave it for future researches in addition to footnote 17.

always satisfied with the second-order condition. As given in equation (26), denote by  $\alpha_i$  the  $i$ -th individual's 'ability' of a maximum amount payable to the government, which is less than or equal to the pre-tax income  $y_i$  or  $\alpha_i \leq y_i$ , then some closed-form solutions have told us in corollaries 1 through 3 and so on as follows:

- [1] that in  $n$  affine labor markets, the  $i$ -th individual ought to pay the optimal income tax  $T_i^*$  for the gross tax revenue  $R$  at a weighted 'ability' portion  $\frac{\alpha_i}{\sum_{i=1}^n \alpha_i}$  or at a ratio of the 'ability'  $\alpha_i$  of the  $i$ -th individual to the sum of 'abilities' of those  $n$  individuals as  $\sum_{i=1}^n \alpha_i$ : That is,  $T_i^* = \frac{\alpha_i}{\sum_{i=1}^n \alpha_i} R$ .
- [1'] that if  $\alpha_i = y_i$  like the other two markets, then the  $i$ -th individual should pay the optimal income tax  $T_i^*$  for the gross tax revenue  $R$  at a weighted 'ability' portion  $\frac{y_i}{\bar{y}}$  or at a ratio of the pre-tax income  $y_i$  of the  $i$ -th individual to the sum of  $n$  individuals' incomes of  $\bar{y} = \sum_{i=1}^n y_i$ : That is,  $T_i^* = \frac{y_i}{\sum_{i=1}^n y_i} R$ .
- [2] that the government in the case of [1] is able to raise her tax revenue  $R$  in descending order of the biggest 'ability'  $\alpha_i$  from among  $n$  individuals until a weighted 'ability' portion is negligible.
- [2'] that the government in the case of [1'] is also able to collect her tax revenue  $R$  in descending order of the biggest income  $y_i$  from among  $n$  individuals till a weighted income portion is sufficiently small.
- [3] that the libertarian 'equal marginal sacrifice' in equation (15) of our proposition 1 would not "involve lopping off the tops of all incomes above the minimum income and leaving everybody, after taxation, with equal incomes (Pigou, 1949, p.57)."
- [4] that the libertarian 'equal marginal sacrifice' of our proposition 1 would not "alter the inequality of incomes by taxation (Dalton, 1967, p.63)."
- [5] that under such a libertarian government, everyone would progressively have an incentive to work harder than before, *etc.*

Consequently, it is interesting to see that the results derived by utilitarians, who believe one of the highest utilities is the best, are totally different from those by libertarians, who believe a 'pure competition' market is the best as shown in footnote 11. It is also interesting to see that some of our libertarian results are akin to the Adam Smith's maxim as shown in footnote 12 where the tax-payment  $T_i$  becomes  $T_i = \frac{y_i}{\sum_{i=1}^n y_i} R$  by  $T_i = \hat{\pi} y_i$  in the proportion of  $\hat{\pi} = \frac{R}{\sum_{i=1}^n y_i}$  as the  $n$  individuals or "subjects of every state ought to contribute towards the support of the government, as nearly as possible, in proportion to their respective abilities; that is, in proportion to the revenue which they respectively enjoy under the protection of the state." So, we can implicate from them that a libertarian government, who believes a 'pure competition' market of labor is the best, may intervene the  $n$  individuals' labor markets and levy taxes  $T_i$  on their pre-tax incomes  $y_i$  ( $\geq \alpha_i$ ) in proportion to their respective incomes  $\frac{y_i}{\sum_{i=1}^n y_i}$ ; as nearly as possible, in proportion to their precise abilities  $\frac{\alpha_i}{\sum_{i=1}^n \alpha_i}$  if she can, and from equations (7) and (8) that such a government may let those  $n$  individuals work for her revenue  $R$  as patrons, as equally as possible, in proportion to their last moments of labor quantities  $\frac{\bar{x}_i - x_i}{\bar{x}_i}$  ( $\equiv t_i = \frac{R}{\sum_{i=1}^n y_i} = \hat{\pi}$ ) for them to contribute toward the support of the libertarian government.

# Appendixes

## A. Proofs

In this appendix A, we prove our proposition and its corollaries in the order of appearance.

☛ POOF OF PROPOSITION 1:

*Proof.* From equation (13), we have

$$(A.1) \quad -\frac{D_i(x_i) - S_i(x_i)}{\bar{w}_i} = \ell \text{ for } i = 1, 2, \dots, n.$$

In the meantime, the derivative of market-surplus  $ms_i$  and that of tax-payment  $T_i$  are given as follows:

$$(A.2) \quad \frac{dms_i}{dx_i} = D_i(x_i) - S_i(x_i) \geq 0, \quad 0 < x_i \leq \bar{x}_i;$$

$$(A.3) \quad \frac{dT_i}{dx_i} = -\bar{w}_i < 0, \quad 0 < x_i \leq \bar{x}_i$$

from equations (10) and (7), respectively. By the chain-rule, we can convey equation (A.1) into

$$(A.4) \quad \frac{dms_i}{dT_i} = \frac{dms_i}{dx_i} \frac{dx_i}{dT_i} = \frac{dms_i}{dx_i} \frac{1}{dT_i/dx_i} = -\frac{dms_i/dx_i}{\bar{w}_i} = \ell \leq 0 \text{ for } i = 1, 2, \dots, n.$$

which is nothing but equation (15). *Q. E. D.* □

☛ POOF OF COROLLARY 1:

*Proof.* Substitute the  $i$ -th market's post-tax equilibrium labor quantity  $x_i = \frac{\alpha_i^A}{\bar{w}_i^A} \ell + \bar{x}_i^A$  in equation (25) into that in equation (14) of

$$(A.5) \quad R = \bar{y}^A - \sum_{i=1}^n \bar{w}_i^A x_i,$$

where  $\bar{y}^A$  is an aggregated level of  $y_i^A = \bar{w}_i^A \times \bar{x}_i^A$  in equation (23) or  $\bar{y}^A \equiv \sum_{i=1}^n y_i^A$ . Then, we have had

$$(A.6) \quad R = \bar{y}^A - \sum_{i=1}^n \bar{w}_i^A \left( \frac{\alpha_i^A}{\bar{w}_i^A} \ell + \bar{x}_i^A \right) = -\ell \sum_{i=1}^n \alpha_i^A$$

and so that an optimal Lagrange-multiplier, say  $\ell^{A*}$  can be solved in a parametric closed-form as

$$(A.7) \quad \ell^{A*} \equiv -\frac{R}{\sum_{i=1}^n \alpha_i^A}.$$

Thus, equations (25) and (A.7) yield each market's optimal post-tax equilibrium quantity of labor  $x_i^{A*}$  as

$$(A.8) \quad x_i^{A*} \equiv -\frac{\alpha_i^A}{\bar{w}_i^A} \frac{R}{\sum_{i=1}^n \alpha_i^A} + \bar{x}_i^A$$

in terms of parameters initially given in our model. Let  $T_i^{A*}$  be an optimal tax-payment in this case A of  $n$  linear markets, and substitute the above equation (A.8) into equation (7) of  $T_i^{A*} = y_i^A - \bar{w}_i^A x_i^{A*}$ , then we finally obtain equation (27). *Q. E. D.* □



⇨⇨ POOF OF COROLLARY 2:

*Proof.* Substitute the  $i$ -th market's post-tax equilibrium labor quantity  $x_i = \frac{\ell + \sqrt{\ell^2 + 4}}{2} \frac{\alpha_i^B}{\bar{w}_i^B}$  in equation (42) into that in equation (14) of

$$(A.9) \quad R = \bar{y}^B - \sum_{i=1}^n \bar{w}_i^B x_i,$$

where  $\bar{y}^B$  is a sum of  $y_i^B = \bar{w}_i^B \times \bar{x}_i^B = \alpha_i^B$  in equations (40) and (43) or  $\bar{y}^B \equiv \sum_{i=1}^n y_i^B$ . Then, we have

$$(A.10) \quad R = \bar{y}^B - \sum_{i=1}^n \bar{w}_i^B \left( \frac{\ell + \sqrt{\ell^2 + 4}}{2} \frac{\alpha_i^B}{\bar{w}_i^B} \right) = \bar{y}^B - \frac{\ell + \sqrt{\ell^2 + 4}}{2} \bar{y}^B$$

and so that an optimal Lagrange-multiplier, say  $\ell^{B*}$  can be solved in a parametric closed-form as

$$(A.11) \quad \ell^{B*} \equiv \frac{(R - 2\bar{y}^B)R}{\bar{y}^B(\bar{y}^B - R)} < 0 \text{ due to that } 0 < R < \bar{y}^B.$$

Hence, equations (42) and (A.11) yield each market's optimal post-tax equilibrium quantity of labor  $x_i^{B*}$  as

$$(A.12) \quad x_i^{B*} \equiv \frac{\bar{y}^B - R}{\bar{y}^B} \bar{x}_i^B$$

in terms of parameters initially given in our model. Let  $T_i^{B*}$  be an optimal tax-payment in this case  $B$  of  $n$  linear markets, and substitute the above equation (A.12) into equation (7) of  $T_i^{B*} = y_i^B - \bar{w}_i^B x_i^{B*}$ , then we obtain equation (44) at last. *Q. E. D.*  $\square$

⇨⇨⇨ POOF OF COROLLARY 3:

*Proof.* Substitute the  $i$ -th market's post-tax equilibrium labor quantity  $x_i = \frac{\alpha_i^C}{\bar{w}_i^C} (\ell + 1)^{1/p}$  in equation (60) into that in equation (14) of

$$(A.13) \quad R = \bar{y}^C - \sum_{i=1}^n \bar{w}_i^C x_i,$$

where  $\bar{y}^C$  is a sum of  $y_i^C = \bar{w}_i^C \times \bar{x}_i^C = \alpha_i^C$  in equations (58) and (61) or  $\bar{y}^C \equiv \sum_{i=1}^n y_i^C$ . Then, we have

$$(A.14) \quad R = \bar{y}^C - \sum_{i=1}^n \bar{w}_i^C \left\{ \frac{\alpha_i^C}{\bar{w}_i^C} (\ell + 1)^{1/p} \right\} = \bar{y}^C - (\ell + 1)^{1/p} \bar{y}^C$$

and so that an optimal Lagrange-multiplier  $\ell^{C*}$  can be solved in a parametric closed-form as

$$(A.15) \quad \ell^{C*} \equiv \left( \frac{\bar{y}^C - R}{\bar{y}^C} \right)^p - 1 < 0 \text{ owing to that } 0 < R < \bar{y}^C.$$

So, equations (60) and (A.15) make up each market's optimal post-tax equilibrium labor quantity  $x_i^{C*}$  as

$$(A.16) \quad x_i^{C*} \equiv \frac{\bar{y}^C - R}{\bar{y}^C} \bar{x}_i^C$$

in terms of parameters initially given in our model. Let  $T_i^{C*}$  be an optimal tax-payment in this case  $C$  of  $n$  linear markets, and substitute the above equation (A.16) into equation (7) of  $T_i^{C*} = y_i^C - \bar{w}_i^C x_i^{C*}$ , then we immediately obtain equation (62). *Q. E. D.*  $\square$

## B. A Bordered Hessian

In this appendix B, we discuss the second-order condition for the Lagrange-function  $\mathcal{L}$  in equation (12) as well as a bordered Hessian  $|\bar{H}|$  such that

$$(B.1) \quad |\bar{H}| \equiv \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & \mathcal{L}_{11} & 0 & \cdots & 0 \\ g_2 & 0 & \mathcal{L}_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & 0 & 0 & \cdots & \mathcal{L}_{nn} \end{vmatrix}$$

including its successive bordered principal minors as

$$(B.2) \quad |\bar{H}_2| \equiv \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & \mathcal{L}_{11} & 0 \\ g_2 & 0 & \mathcal{L}_{22} \end{vmatrix}, |\bar{H}_3| \equiv \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & \mathcal{L}_{11} & 0 & 0 \\ g_2 & 0 & \mathcal{L}_{22} & 0 \\ g_3 & 0 & 0 & \mathcal{L}_{33} \end{vmatrix}, \cdots, |\bar{H}_n|,$$

the last of which must be exactly the same as equation (B.1) or  $|\bar{H}_n| \equiv |\bar{H}|$  whose non-zero components of partial derivatives  $g_i$  and  $\mathcal{L}_{ii}$  are supposed to be negative in equations (16) and (17): We rewrite them as

$$(B.3) \quad g_i \equiv \frac{\partial g}{\partial x_i} = \frac{d}{dx_i} g = \frac{d}{dx_i} (\bar{y} - \sum_{i=1}^n \bar{w}_i x_i) = -\bar{w}_i < 0;$$

$$(B.4) \quad \mathcal{L}_{ii} \equiv \frac{\partial^2 \mathcal{L}}{\partial x_i^2} = \frac{d}{dx_i} \left( \frac{\partial \mathcal{L}}{\partial x_i} \right) = \frac{d}{dx_i} \{D_i(x_i) - S_i(x_i) + \ell \bar{w}_i\} = D'_i(x_i) - S'_i(x_i) < 0$$

from equations (1) and (2):  $S' > 0$ ;  $D' \leq 0$ , respectively. Then, principal minors in equation (B.2) alternately become positive and negative as  $|\bar{H}_2| = -\mathcal{L}_{22} g_1^2 - \mathcal{L}_{11} g_2^2 > 0$ ,  $|\bar{H}_3| = \mathcal{L}_{33} |\bar{H}_2| - \mathcal{L}_{11} \mathcal{L}_{22} g_3^2 < 0$ ,  $|\bar{H}_4| = \mathcal{L}_{44} |\bar{H}_3| - \mathcal{L}_{11} \mathcal{L}_{22} \mathcal{L}_{33} g_4^2 > 0$ ,  $\cdots$ , and so provided that  $|\bar{H}_n| = (-1)^n |\bar{H}_n| > 0$  as stated in Alpha C. Chiang (1984, p.385). Now, we are always able to induce further that  $|\bar{H}_{n+1}| = \mathcal{L}_{n+1} |\bar{H}_n| - \mathcal{L}_{11} \mathcal{L}_{22} \cdots \mathcal{L}_{nn} g_{n+1}^2 < 0$  because of equations (B.3) and (B.4):  $g_i < 0$ ;  $\mathcal{L}_{ii} < 0$  for  $i = 1, 2, \cdots, n$ , and  $n + 1$ .

Therefore, all we have to do here for the second-order condition corresponding to an optimal income-tax problem like equation (12) is to examine if all non-zero elements of the bordered Hessian  $|\bar{H}|$  in equation (B.1), or partial derivatives  $g_i$  and  $\mathcal{L}_{ii}$  are negative:  $g_i < 0$ ;  $\mathcal{L}_{ii} < 0$  for  $i = 1, 2, \cdots, n$ .

### C. The principle of ‘equal sacrifice’ of utility

Based upon Hugh Dalton (1967, p.68), we would like to show in this appendix C that a functional relation between utility  $u$  and income  $y$  is able to give us a few of formulae for the principle of ‘equal sacrifice,’ in which “the direct money burden of taxation should be so distributed that the direct real burden on all taxpayers is equal” as put in Dalton (*ibid.*, p.63).

➤C① In the case of a utility function  $u_1$  under the Daniel Bernoulli’s Law: Suppose that there were a government that imposed a tax rate  $t_i$  on incomes of  $n$  individuals, each of who had an income  $y_i > 0$  with common utility function  $u_1 \equiv \ln y_i$  where  $i$  is a running index as  $i = 1, 2, \cdots, n$ . Besides, denote by  $z_i$  a disposable income of the  $i$ -th individual after tax as  $z_i \equiv y_i - T_i = y_i - t_i \times y_i = (1 - t_i)y_i$ . According to the principle of ‘equal sacrifice,’ we must have the ‘sacrifice’ of  $\ln y_i - \ln z_i$  been equally some constant, say  $\kappa_1 > 0$  here such that

$$(C.1) \quad \kappa_1 \equiv \ln y_i - \ln z_i = \ln y_i - \ln\{(1 - t_i)y_i\} = -\ln(1 - t_i) > 0,$$

which yields to us that  $e^{-\kappa_1} = 1 - t_i > 0$  where  $e$  is the base of natural logarithm or  $e \approx 2.71828 \dots$ . Then, the tax rate  $t_i$  should be constant for all  $i$  as

$$(C.2) \quad t_i = 1 - e^{-\kappa_1} > 0; \quad t_i < 1, \quad \kappa_1 > 0.$$

Thus, the  $i$ -th individual pays  $T_i$  for the tax to keep  $z_i$  as the disposable income as follows:

$$(C.3) \quad T_i = t_i y_i = (1 - e^{-\kappa_1})y_i > 0, \quad y_i > 0, \quad \kappa_1 > 0;$$

$$(C.4) \quad z_i = y_i - T_i = e^{-\kappa_1}y_i > 0, \quad y_i > 0, \quad \kappa_1 > 0$$

for all  $i = 1, 2, \dots, n$ . Then, the government is able to obtain her tax revenue, say  $R_1$  by

$$(C.5) \quad R_1 \equiv \sum_{i=1}^n T_i = \sum_{i=1}^n t_i y_i = (1 - e^{-\kappa_1})\bar{y} > 0, \quad y_i > 0, \quad \kappa_1 > 0,$$

where  $\bar{y}$  is an aggregated income level of  $\bar{y} \equiv \sum_{i=1}^n y_i$ .<sup>20</sup> So, we can see here that although a rate of change of the  $i$ -th individual's tax rate  $t_i$  with respect to the income level  $y_i$  is nil as

$$(C.6) \quad \frac{dt_i}{dy_i} = 0$$

from equation (C.2), not only that of the tax-payment  $T_i$  but also that of the disposable income  $z_i$  after tax is a positive constant as

$$(C.7) \quad \frac{dT_i}{dy_i} = 1 - e^{-\kappa_1} > 0 \quad \text{with} \quad \frac{d^2 T_i}{dy_i^2} = 0$$

from equation (C.3) and as

$$(C.8) \quad \frac{dz_i}{dy_i} = e^{-\kappa_1} > 0 \quad \text{with} \quad \frac{d^2 z_i}{dy_i^2} = 0$$

from equation (C.4), respectively.  $\square$

**➔C②** In the case of a utility function  $u_2$  of Dalton (*ibid.*, p.69): Other things being equal, let us replace  $u_1$  by  $u_2 \equiv c - \frac{1}{y_i}$ ,  $y_i > 0$  where  $c$  is a constant, we should have the following constant 'sacrifice'  $\kappa_2 > 0$  of

$$(C.9) \quad \kappa_2 \equiv c - \frac{1}{y_i} - \left(c - \frac{1}{z_i}\right) = -\frac{1}{y_i} + \frac{1}{(1 - t_i)y_i} = \frac{t_i}{(1 - t_i)y_i} > 0,$$

which implies that  $\kappa_2(1 - t_i)y_i = t_i$ , or that  $\kappa_2 y_i = t_i + \kappa_2 t_i y_i = t_i(1 + \kappa_2 y_i)$ . So, the tax rate  $t_i$  becomes

$$(C.10) \quad t_i = \frac{\kappa_2 y_i}{1 + \kappa_2 y_i} = 1 - \frac{1}{1 + \kappa_2 y_i} > 0; \quad t_i < 1, \quad y_i > 0, \quad \kappa_2 > 0.$$

Thus, the  $i$ -th individual pays  $T_i$  for the tax to have the disposable income  $z_i$  as follows:

$$(C.11) \quad T_i = t_i y_i = \frac{\kappa_2 y_i^2}{1 + \kappa_2 y_i} = \left(1 - \frac{1}{1 + \kappa_2 y_i}\right)y_i > 0, \quad y_i > 0, \quad \kappa_2 > 0;$$

<sup>20</sup>It is interesting to see from equation (C.5) that a value  $\kappa_1$  can be easily estimated as  $\hat{\kappa}_1 = \ln\left(\frac{\bar{y}}{\bar{y} - R_1}\right)$  by  $1 - e^{-\kappa_1} = \frac{R_1}{\bar{y}}$ .

$$(C.12) \quad z_i = y_i - T_i = \frac{y_i}{1 + \kappa_2 y_i} > 0, \quad y_i > 0, \quad \kappa_2 > 0$$

for all  $i = 1, 2, \dots, n$ , respectively. From equation (C.11), the government can obtain her tax revenue, say  $R_2$  this time as

$$(C.13) \quad R_2 \equiv \sum_{i=1}^n T_i = \sum_{i=1}^n \frac{\kappa_2 y_i^2}{1 + \kappa_2 y_i} = \bar{y} - \sum_{i=1}^n \frac{y_i}{1 + \kappa_2 y_i} > 0, \quad y_i > 0, \quad \kappa_2 > 0,$$

where  $\bar{y}$  is the aggregated income of  $\bar{y} \equiv \sum_{i=1}^n y_i$ .<sup>21</sup> So, it can be easily seen in the Dalton's case that a rate of change of the  $i$ -th individual's tax rate  $t_i$  with respect to the income level  $y_i$  is decreasingly progressive as

$$(C.14) \quad \frac{dt_i}{dy_i} = \frac{\kappa_2}{(1 + \kappa_2 y_i)^2} > 0 \quad \text{and} \quad \frac{d^2 t_i}{dy_i^2} = -\frac{2\kappa_2^2}{(1 + \kappa_2 y_i)^3} < 0$$

from equation (C.10). So is that of the disposable income  $z_i$  after tax as

$$(C.15) \quad \frac{dz_i}{dy_i} = \frac{1}{(1 + \kappa_2 y_i)^2} > 0 \quad \text{and} \quad \frac{d^2 z_i}{dy_i^2} = -\frac{2\kappa_2}{(1 + \kappa_2 y_i)^3} < 0$$

from equation (C.12). However, that of the tax-payment  $T_i$  increasingly progressive as

$$(C.16) \quad \frac{dT_i}{dy_i} = \frac{(2 + \kappa_2 y_i)\kappa_2 y_i}{(1 + \kappa_2 y_i)^2} > 0 \quad \text{and} \quad \frac{d^2 T_i}{dy_i^2} = \frac{2\kappa_2}{(1 + \kappa_2 y_i)^3} > 0$$

from equation (C.11).  $\square$

#### D. The principle of 'proportional sacrifice' of utility

Similarly to appendix C, in this appendix D, we would like to show some formulas for the principle of 'proportional sacrifice,' in which "the direct real burden on taxation should be so distributed that the direct real burden on all every taxpayer is proportionate to the economic welfare which he derives from his income" as mentioned in Dalton (*ibid.*, p.63).

⇒D① In the case of the Bernoulli's utility function  $u_1$ : Suppose again there were a government that levied a tax rate  $t_i$  on incomes of  $n$  individuals, each of who had an income  $y_i > 0$  with common utility function  $u_1 \equiv \ln y_i$  where  $i$  is a running index as  $i = 1, 2, \dots, n$ . Let  $z_i$  be a disposable income of the  $i$ -th individual after tax as  $z_i \equiv y_i - T_i = y_i - t_i \times y_i = (1 - t_i)y_i$ . According to the principle of 'proportional sacrifice,' we must have a certain constant rate  $\kappa_3$ ;  $0 < \kappa_3 < 1$  such that with  $\kappa_1$  of equation (C.1),

$$(D.1) \quad 0 < \kappa_3 \equiv \frac{\kappa_1}{\ln y_i} = -\frac{\ln(1 - t_i)}{\ln y_i} < 1,$$

which gives us that  $-\kappa_3 \ln y_i = \ln(1 - t_i)$  or  $y_i^{-\kappa_3} = 1 - t_i > 0$ . Then, the tax rate  $t_i$  turns to be

$$(D.2) \quad t_i = 1 - y_i^{-\kappa_3} \begin{cases} \geq 0; \quad t_i < 1 & \text{if } y_i \geq 1 \text{ and so that } y_i^{\kappa_3} \geq 1, \quad 0 < \kappa_3 < 1, \\ < 0 & \text{if } 0 < y_i < 1 \text{ and so that } 0 < y_i^{\kappa_3} < 1, \quad 0 < \kappa_3 < 1 \end{cases}$$

<sup>21</sup>Differently from  $\hat{\kappa}_1$  in footnote 20, it does not seem to be so easy to estimate a value  $\kappa_2$  in equation (C.13) that by assuming all  $n$  individuals had an identical pre-tax income  $\bar{y}$  or an average  $\bar{y}$  of  $\bar{y}$  such that  $n\bar{y} = \bar{y} \equiv \sum_{i=1}^n y_i$ , one may estimate its  $\kappa_2$  as  $\hat{\kappa}_2 \approx \frac{R_2}{(\bar{y} - R_2)\bar{y}}$  from the gross revenue of  $R_2 = \frac{n\kappa_2\bar{y}^2}{1 + \kappa_2\bar{y}}$  in equation (C.13).

for all  $i$ . Hence, the  $i$ -th individual pays  $T_i$  for the tax to keep  $z_i$  as the disposable income as follows:

$$(D.3) \quad T_i = t_i y_i = (1 - y_i^{-\kappa_3}) y_i \begin{cases} \geq 0; & t_i < 1 & \text{if } y_i \geq 1 \text{ and so that } y_i^{\kappa_3} \geq 1, \quad 0 < \kappa_3 < 1, \\ < 0 & & \text{if } 0 < y_i < 1 \text{ and so that } 0 < y_i^{\kappa_3} < 1, \quad 0 < \kappa_3 < 1; \end{cases}$$

$$(D.4) \quad z_i = y_i - T_i = y_i^{1-\kappa_3} > 0, \quad y_i > 0, \quad 0 < \kappa_3 < 1$$

for all  $i = 1, 2, \dots, n$ . The government can collect her tax revenue, say  $\underline{R}_1$  including a negative tax as a transfer payment of

$$(D.5) \quad \underline{R}_1 \equiv \sum_{i=1}^n T_i = \bar{y} - \sum_{i=1}^n y_i^{1-\kappa_3} > 0, \quad y_i > 0, \quad 0 < \kappa_3 < 1,$$

in which  $\bar{y}$  is the aggregated income level of  $\bar{y} \equiv \sum_{i=1}^n y_i$ .<sup>22</sup> It is easy to see that a rate of change of the  $i$ -th individual's tax rate  $t_i$  with respect to the income level  $y_i$  is decreasingly progressive as

$$(D.6) \quad \frac{dt_i}{dy_i} = \kappa_3 y_i^{-\kappa_3-1} > 0 \text{ and } \frac{d^2 t_i}{dy_i^2} = -\kappa_3(\kappa_3 + 1) y_i^{-\kappa_3-1} < 0$$

from equation (D.2). So is that of the disposable income  $z_i$  as

$$(D.7) \quad \frac{dz_i}{dy_i} = (1 - \kappa_3) y_i^{-\kappa_3} > 0 \text{ and } \frac{d^2 z_i}{dy_i^2} = -\kappa_3(1 - \kappa_3) y_i^{-\kappa_3-1} < 0$$

from equation (D.4). However, that of the tax-payment  $T_i$  is increasingly progressive almost everywhere as

$$(D.8a) \quad \frac{dT_i}{dy_i} = 1 - (1 - \kappa_3) y_i^{-\kappa_3} > 0 \text{ with } \frac{d^2 T_i}{dy_i^2} = \kappa_3(1 - \kappa_3) y_i^{-\kappa_3-1} > 0$$

if  $y_i^{\kappa_3} + \kappa_3 > 1$ ,  $y_i^{\kappa_3} > 0$ , and  $0 < \kappa_3 < 1$  from equation (D.3) whereas decreasingly regressive as

$$(D.8b) \quad \frac{dT_i}{dy_i} = 1 - (1 - \kappa_3) y_i^{-\kappa_3} < 0 \text{ with } \frac{d^2 T_i}{dy_i^2} = \kappa_3(1 - \kappa_3) y_i^{-\kappa_3-1} > 0$$

only if  $0 < y_i^{\kappa_3} + \kappa_3 < 1$ ,  $0 < y_i^{\kappa_3} < 1$ , and  $0 < \kappa_3 < 1$ .  $\square$

$\Rightarrow$ D② In the case of a utility function  $u_2$  of Dalton (*ibid.*, p.69): Other things being equal, let us replace  $u_1$  by  $u_2 \equiv c - \frac{1}{y_i}$ ,  $y_i > 0$  where  $c$  is a constant. We have to have the following 'proportional sacrifice' of a certain constant rate  $\kappa_4$ ;  $0 < \kappa_4 < 1$  such that from  $\kappa_2$  in equation (C.9),

$$(D.9) \quad 0 < \kappa_4 \equiv \frac{\kappa_2}{c - 1/y_i} = \frac{t_i}{(1 - t_i)y_i} \times \frac{y_i}{cy_i - 1} = \frac{t_i}{(1 - t_i)(cy_i - 1)} < 1,$$

which yields to us that  $\kappa_4(1 - t_i)(cy_i - 1) = t_i$ , or  $\kappa_4(cy_i - 1) = t_i + \kappa_4 t_i(cy_i - 1) = t_i\{1 + \kappa_4(cy_i - 1)\}$ .

Then, the tax rate  $t_i$  is calculated as

$$(D.10) \quad t_i = 1 - \frac{1}{1 - \kappa_4 + c\kappa_4 y_i} \begin{cases} \geq 0; & t_i < 1 & \text{if } y_i \geq 1/c > 0, \quad c > 0, \quad 0 < \kappa_4 < 1, \\ < 0 & & \text{if } 0 < y_i < 1/c, \quad c > 0, \quad 0 < \kappa_4 < 1. \end{cases}$$

<sup>22</sup>Differently from footnote 21, there is a negative tax  $T_i$  in equation (D.3) or (D.5). However, assuming that all  $n$  individuals had the identical pre-tax income  $\bar{y}$  or the average  $\bar{y}$  of  $\bar{y}$  such that  $n\bar{y} = \bar{y}$ , one might estimate a value  $\kappa_3$  as  $\hat{\kappa}_3 \approx \frac{\ln \bar{y} - \ln(\bar{y} - \underline{R}_1)}{\ln \bar{y}}$  from the net revenue  $\underline{R}_1 \approx \bar{y} - n\bar{y}^{1-\kappa_3}$  in equation (D.5).

Thus, the  $i$ -th individual pays  $T_i$  for the tax to have the disposable income  $z_i$  as follows:

$$(D.11) \quad T_i = t_i y_i = \left(1 - \frac{1}{1 - \kappa_4 + c \kappa_4 y_i}\right) y_i \begin{cases} \geq 0; t_i < 1 & \text{if } y_i \geq 1/c > 0, c > 0, 0 < \kappa_4 < 1, \\ < 0 & \text{if } 0 < y_i < 1/c, c > 0, 0 < \kappa_4 < 1; \end{cases}$$

$$(D.12) \quad z_i = y_i - T_i = \frac{y_i}{1 - \kappa_4 + c \kappa_4 y_i} > 0, y_i > 0, c > 0, 0 < \kappa_4 < 1$$

for all  $i = 1, 2, \dots, n$ , respectively. Denote by  $\bar{y}$  the aggregated income as  $\bar{y} \equiv \sum_{i=1}^n y_i$ , and the government's tax revenue, say  $\underline{R}_2$ , which contains a negative tax as a transfer payment, can be computed as follows:<sup>23</sup>

$$(D.13) \quad \underline{R}_2 \equiv \sum_{i=1}^n T_i = \bar{y} - \sum_{i=1}^n \frac{y_i}{1 - \kappa_4 + c \kappa_4 y_i} > 0, y_i > 0, c > 0, 0 < \kappa_4 < 1.$$

It is now easy to see that a rate of change of the  $i$ -th individual's tax rate  $t_i$  with respect to the income level  $y_i$  is decreasingly progressive as

$$(D.14) \quad \frac{dt_i}{dy_i} = \frac{c \kappa_4}{(1 - \kappa_4 + c \kappa_4 y_i)^2} > 0 \text{ and } \frac{d^2 t_i}{dy_i^2} = -\frac{2c^2 \kappa_4^2}{(1 - \kappa_4 + c \kappa_4 y_i)^3} < 0$$

from equation (D.10). From equation (D.12), so is that of the disposable income  $z_i$  as

$$(D.15) \quad \frac{dz_i}{dy_i} = \frac{1 - \kappa_4}{(1 - \kappa_4 + c \kappa_4 y_i)^2} > 0 \text{ and } \frac{d^2 z_i}{dy_i^2} = -\frac{2c \kappa_4 (1 - \kappa_4)}{(1 - \kappa_4 + c \kappa_4 y_i)^3} < 0.$$

From equation (D.11), however, that of the tax-payment  $T_i$  increasingly progressive almost everywhere as

$$(D.16a) \quad \frac{dT_i}{dy_i} = 1 - \frac{1 - \kappa_4}{(1 - \kappa_4 + c \kappa_4 y_i)^2} > 0 \text{ with } \frac{d^2 T_i}{dy_i^2} = \frac{2c \kappa_4 (1 - \kappa_4)}{(1 - \kappa_4 + c \kappa_4 y_i)^3} > 0$$

if  $y_i > \frac{\sqrt{1 - \kappa_4} - (1 - \kappa_4)}{c \kappa_4} > 0$  and  $0 < \kappa_4 < 1$ , but decreasingly regressive as

$$(D.16b) \quad \frac{dT_i}{dy_i} = 1 - \frac{1 - \kappa_4}{(1 - \kappa_4 + c \kappa_4 y_i)^2} < 0 \text{ with } \frac{d^2 T_i}{dy_i^2} = \frac{2c \kappa_4 (1 - \kappa_4)}{(1 - \kappa_4 + c \kappa_4 y_i)^3} > 0$$

only if  $0 < y_i < \frac{\sqrt{1 - \kappa_4} - (1 - \kappa_4)}{c \kappa_4}$  and  $0 < \kappa_4 < 1$ .  $\square$

### E. The principle of 'equal marginal sacrifice' and the Pigowian transfer

In this appendix E, we would like to employ two utility functions of Daniel Bernoulli as well as Hugh Dalton again for showing that such a utilitarian principle of 'equal marginal sacrifice,' in which "the total direct real burden on the taxpayers as a whole is as small as possible" as described in Dalton (1967, p.63), "would involve lopping off the tops of all incomes above the minimum income and leaving everybody, after taxation, with equal incomes" as stated in Arthur C. Pigou (1949, p.57).

➤E① In the case of the utility Function  $u_1$  under the Bernoulli's Law: Suppose that there were  $n$  individuals, each of who had a pre-tax income  $y_i$  with common utility function  $u_1 \equiv \ln z_i$  where  $z_i$  is a post-tax income

<sup>23</sup>Assuming similarly to footnotes 21 and 22 that all  $n$ -individual had the identical pre-tax income  $\bar{y}$  or the average  $\bar{y}$  of  $\bar{y}$  such that  $n \bar{y} = \bar{y}$ , one might determine a value  $\kappa_4$  as  $\hat{\kappa}_4 \approx \frac{\underline{R}_2}{(\bar{y} - \underline{R}_2)(c \bar{y} - 1)}$  from the net revenue  $\underline{R}_2 = \bar{y} - \frac{n \bar{y}}{1 - \kappa_4 + c \kappa_4 \bar{y}}$  in equation (D.13).

or disposable one of the  $i$ -th individual and  $i$  is a running index as  $i = 1, 2, \dots, n$ , and so that a government maximized a sum of  $u_1 = \ln z_i \leq \ln y_i$  with respect to tax rates  $t_i$  subject to her tax revenue constraint  $\underline{R}$  of a sum of tax-payment  $T_i \equiv t_i \times y_i$  by the  $i$ -th individual; *n. b.*,  $\underline{R}$  implicitly “denotes the net revenue to be raised” by the government as noted in Anthony B. Atkinson (1973, p.91). Besides, let  $\lambda_1$  be a Lagrange-multiplier, then a Lagrange-function  $\mathcal{L}_1$  becomes

$$(E.1) \quad \mathcal{L}_1 \equiv \sum_{i=1}^n \ln z_i + \lambda_1 (\underline{R} - \sum_{i=1}^n T_i) = \sum_{i=1}^n \ln\{(1 - t_i)y_i\} + \lambda_1 (\underline{R} - \sum_{i=1}^n t_i y_i), \quad 1 - t_i > 0,$$

where the post-tax or disposable income  $z_i$  is expanded into  $z_i \equiv y_i - T_i = y_i - t_i \times y_i = (1 - t_i)y_i$ . From equation (E.1), we have the following first-order condition:

$$(E.2) \quad \frac{\partial \mathcal{L}_1}{\partial t_i} = -\frac{1}{1 - t_i} - \lambda_1 y_i = 0, \quad t_i < 1 \text{ for all } i = 1, 2, \dots, n;$$

$$(E.3) \quad \frac{\partial \mathcal{L}_1}{\partial \lambda_1} = \underline{R} - \sum_{i=1}^n t_i y_i = 0.$$

So, equation (E.2) yields the principle of ‘equal marginal sacrifice’ of utility  $u_1$  with respect to the tax  $T_i$  as

$$(E.4) \quad \frac{d u_1}{d T_i} = \frac{d u_1}{d t_i} \frac{1}{d T_i / d t_i} = -\frac{1}{1 - t_i} \frac{1}{y_i} = \lambda_1 < 0, \quad t_i < 1 \text{ for all } i = 1, 2, \dots, n$$

by the chain-rule of the derivatives  $\frac{d u_1}{d t_i} = -\frac{1}{1 - t_i}$  and  $\frac{d T_i}{d t_i} = y_i$ . Sum up reciprocals of equation (E.4) like

$$(E.5) \quad -\frac{1}{\lambda_1} = (1 - t_i)y_i = y_i - t_i y_i = y_i - T_i = z_i > 0,$$

each of which reflects an identical disposable income after tax, or  $z_1 = z_2 = \dots = z_n$ , then we have that

$$(E.6) \quad -\frac{n}{\lambda_1} = \sum_{i=1}^n y_i - \sum_{i=1}^n t_i y_i = \bar{y} - \underline{R} > 0,$$

where  $\bar{y}$  is an aggregate income of  $\bar{y} \equiv \sum_{i=1}^n y_i$  and  $\underline{R}$  is the net revenue as  $\underline{R} \equiv \sum_{i=1}^n t_i y_i$  in equation (E.3).

By substituting the following optimal Lagrange-multiplier  $\lambda_1^*$  derived from equation (E.6) as

$$(E.7) \quad \lambda_1^* \equiv -\frac{n}{\bar{y} - \underline{R}} < 0$$

back into equation (E.5), hence, we are sequentially able to obtain an optimal disposable income, say  $z_i^* \equiv -\frac{1}{\lambda_1^*}$  identically to all  $z_1^* = z_2^* = \dots = z_n^*$ , an optimal tax-payment  $T_i^*$ , and an optimal tax rate  $t_i^*$  as follows:

$$(E.8) \quad z_i^* \equiv \frac{1}{n}(\bar{y} - \underline{R}) > 0;$$

$$(E.9) \quad T_i^* \equiv y_i - z_i^* = y_i - \frac{1}{n}(\bar{y} - \underline{R}) \begin{cases} \geq 0 & \text{if } y_i \geq (\bar{y} - \underline{R})/n, \\ < 0 & \text{if } 0 < y_i < (\bar{y} - \underline{R})/n; \end{cases}$$

$$(E.10) \quad t_i^* \equiv \frac{T_i^*}{y_i} = 1 - \frac{z_i^*}{y_i} = 1 - \frac{1}{n y_i}(\bar{y} - \underline{R}) \begin{cases} \geq 0; t_i^* < 1 & \text{if } y_i \geq (\bar{y} - \underline{R})/n, \\ < 0 & \text{if } 0 < y_i < (\bar{y} - \underline{R})/n \end{cases}$$

for all  $i = 1, 2, \dots, n$ , respectively.

As discussed in footnote 15, consequently, some cross-sectional studies to equations (E.8) through (E.10), in which the aggregated pre-tax income  $\bar{y} \equiv \sum_{i=1}^n y_i$  is always fixed as a constant, are able to tell us that

$$(E.11) \quad \frac{dz_i^*}{dy_i} = 0 \text{ and } \frac{d^2 z_i^*}{dy_i^2} = 0;$$

$$(E.12) \quad \frac{dT_i^*}{dy_i} = 1 > 0 \text{ and } \frac{d^2 T_i^*}{dy_i^2} = 0;$$

$$(E.13) \quad \frac{dt_i^*}{dy_i} = \frac{z_i^*}{y_i^2} > 0 \text{ and } \frac{d^2 t_i^*}{dy_i^2} = -\frac{2z_i^*}{ny_i^3} < 0;$$

on the other hand, some ordinary comparative-static analyses to equations (E.8) through (E.10) can give us with another running index  $r$  used in footnote 13 that for all  $r = 1, 2, \dots, n$ ,

$$(E.14) \quad \frac{dz_r^*}{dy_r} = \frac{1}{n} > 0 \text{ and } \frac{d^2 z_r^*}{dy_r^2} = 0;$$

$$(E.15) \quad \frac{dT_r^*}{dy_r} = 1 - \frac{1}{n} > 0 \text{ and } \frac{d^2 T_r^*}{dy_r^2} = 0;$$

$$(E.16) \quad \frac{dt_r^*}{dy_r} = \frac{\bar{y} - y_r - \underline{R}}{ny_r^2} > 0 \text{ and } \frac{d^2 t_r^*}{dy_r^2} = -\frac{2(\bar{y} - y_r - \underline{R})}{ny_r^3} < 0.$$

So, it is interesting to see in equations (E.8) and (E.11) that the utilitarian ‘equal marginal sacrifice’ “would involve lopping off the tops of all incomes above the minimum income and leaving everybody, after taxation, with equal incomes” as mentioned in Pigou (1949, p.57), in equation (E.14) that even then somebody might still have an incentive to work for everybody a little bit (by one  $n$ -th), and so on.  $\square$

$\triangleright$ E② In the case of the utility Function  $u_2$  of Dalton (*ibid.*, p.69), or  $u_2 \equiv c - \frac{1}{z_i}$  where  $c$  is a constant and  $z_i$  is the  $i$ -th individual’s post-tax (disposable) income calculated with a pre-tax income  $y_i$ , tax-payment  $T_i$ , and a tax rate  $t_i$  as  $z_i \equiv y_i - T_i = y_i - t_i \times y_i = (1 - t_i)y_i$ : Other things being equal, let us replace  $u_1$  by  $u_2$  as well as  $\lambda_1$  by another multiplier  $\lambda_2$ , then the previous function  $\mathcal{L}_1$  changes into a new one  $\mathcal{L}_2$  of

$$(E.17) \quad \mathcal{L}_2 \equiv \sum_{i=1}^n \left( c - \frac{1}{z_i} \right) + \lambda_2 (\underline{R} - \sum_{i=1}^n T_i) = \sum_{i=1}^n \left\{ c - \frac{1}{(1 - t_i)y_i} \right\} + \lambda_2 (\underline{R} - \sum_{i=1}^n t_i y_i), \quad 1 - t_i > 0,$$

which yields to us the following first-order condition:

$$(E.18) \quad \frac{\partial \mathcal{L}_2}{\partial t_i} = -\frac{1}{(1 - t_i)^2 y_i} - \lambda_2 y_i = 0, \quad t_i < 1 \text{ for all } i = 1, 2, \dots, n;$$

$$(E.19) \quad \frac{\partial \mathcal{L}_2}{\partial \lambda_2} = \underline{R} - \sum_{i=1}^n t_i y_i = 0.$$

Equation (E.18) gives us the principle of ‘equal marginal sacrifice’ of utility  $u_2$  with respect to the tax  $T_i$  as

$$(E.20) \quad \frac{du_2}{dT_i} = \frac{du_2}{dt_i} \frac{1}{dT_i/dt_i} = -\frac{1}{(1 - t_i)^2 y_i} \frac{1}{y_i} = \lambda_2 < 0, \quad t_i < 1 \text{ for all } i = 1, 2, \dots, n$$



by the chain-rule of  $\frac{du_2}{dt_i} = -\frac{1}{(1-t_i)^2 y_i}$  and  $\frac{dT_i}{dt_i} = y_i$ . Add up all positive square-root of a reciprocal of equation (E.20) as

$$(E.21) \quad \sqrt{-1/\lambda_2} = (1-t_i)y_i = y_i - t_i y_i = y_i - T_i = z_i > 0,$$

then we have an optimal Lagrange-multiplier  $\lambda_2^*$  of

$$(E.22) \quad n \sqrt{-1/\lambda_2^*} = \sum_{i=1}^n y_i - \sum_{i=1}^n t_i y_i = \bar{y} - \underline{R} > 0,$$

where  $\bar{y}$  is an aggregate pre-tax income  $\bar{y} \equiv \sum_{i=1}^n y_i$  and  $\underline{R}$  is the net revenue  $\underline{R} \equiv \sum_{i=1}^n t_i y_i$  in equation (E.19) or (E.3). So, substituting equation (E.22) back into (E.21) provides us with exactly the same optimal post-tax income of  $z_i^*$  in equation (E.8) so that all results from equations (E.9) through (E.16) must be available here again for the Dalton's case.  $\square$

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