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Abstract

I introduce a concept of $D(k)$-proofness which says that a rule is nonmanipulable by the false preference relations within k distance from the sincere one. I prove that for every rule defined over all weak orders, strategy-proofness is equivalent to $D(k)$-proofness if and only if $k \geq m - 1$, where m is the number of the alternatives.

Keywords: $D(k)$-proofness, manipulability, strategy-proofness.

JEL classification: D71.

1 Introduction

This paper, following Sato (2010a,b), deals with the question whether the reluctance to make a large lie is helpful for the designer to construct a nonmanipulable rules. Sato (2010b) points out that on the universal domain of *weak orders*, for ev-

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ery $k \in \{1, \ldots, m-2\}$, if the agents misreport only preference within k distance[1] from the true one, where m is the number of the alternatives, then the reluctance to make a large lie is *effective* in the sense that there exists a nonmanipulable rule violating *strategy-proofness*. In other words, for every $k \leq m-2$, the two statements

(i) the agents are free to misreport any preferences, and

(ii) the agents misreport only preferences within k distance from the true preferences

are *not* equivalent on the universal domain of weak orders. This makes a contrast with Sato (2010a) according to which, on the universal domain of *linear orders*, a rule is nonmanipulable by preferences that are adjacent to true ones if and only if it is strategy-proof.

In this paper, I investigate the range of effective extent of the agents' reluctance to make a large lie. For this purpose, I introduce a concept of $D(k)$-*proofness* which says that a rule is nonmanipulable by the false preference relations within k distance from the sincere one. According to this terminology, the result mentioned above can be stated as "for every $k \leq m-2$, $D(k)$-proofness and strategy-proofness are not equivalent".

It is natural to expect that if we increase k to $m-1, m, m+1, \ldots$, then $D(k)$-proofness becomes equivalent to strategy-proofness at some stage. Note that when $D(k)$-proofness and strategy-proofness are equivalent, then agents' reluctance to make a large lie embodied in $D(k)$-proofness is not helpful at all to construct a nonmanipulable rule.

I will prove that, for every $k \geq m-1$, the above two statements (i) and (ii) are equivalent. More precisely, for every $k \geq m-1$ and for every rule, it is non-

[1] The distance between two weak orders is measured by the Kemeny distance (Kemeny, 1959; Kemeny and Snell, 1962) which will be formally introduced in Section 2.

manipulable under the statement (i) if and only if it is nonmanipulable under the statement (ii). Because the statement (i) leads to the Gibbard-Satterthwaite theorem, the statement (ii) also leads to the impossibility result. Thus, together with the result by Sato (2010b) mentioned above, in terms of extent of the agents' reluctance to make a large lie, the number $(m-1)$ is the boundary between effectiveness and ineffectiveness in constructing a nonmanipulable rule.

To illustrate this result, I give an example of a pair of preferences (formulated as weak orders) between which the distance is $(m-1)$. Let R^* denote the weak order under which any two alternatives are indifferent. Let x denote an arbitrary alternative, and consider the weak order R' under which x is strictly preferred to any other alternatives and any two alternatives except x are indifferent. The distance between the two weak orders, R^* and R', is exactly $(m-1)$. Then, it is crucial how agents evaluate the discrepancy between such R^* and R' If they consider the discrepancy is not enough to discourage them to misreport R' with the true preference relation R^*, then their reluctance is "useless" to construct a nonmanipulable rule. On the other hand, if the discrepancy is enough for the agents to refrain from misreporting R' with the true preference relation R^*, then their reluctance is effective.

The paper is organized as follows. In Section 2, I give basic notation, definitions, and preliminary results. In Section 2, I state a main theorem. Section 4 concludes.

2 Definitions and preliminary results

Let $N = \{1, \ldots, n\}$ be a finite set of agents, let X be a finite set of alternatives with $|X| = m \geq 3$.[2] Let $\mathcal{W}$ be the set of all weak orders on X, and let $\mathcal{L}$ denote

[2]For any set A, $|A|$ denotes its cardinality.

the set of all linear orders on X.[3] Elements of $\mathcal{W}$ are *preference relations*. Typical notation for an element of $\mathcal{W}$ is R, and subscripts represent indices of agents; R_i is a preference relation of agent i. The strict parts of R and R_i are represented as P and P_i, and the indifference parts are represented as I and I_i, respectively.

The indifference part I of $R \in \mathcal{W}$ is an equivalence relation,[4] and it induces a partition of X. Let $r_k(R)$ denote the kth indifference class. For example, $r_1(R)$ denotes the set $\{x \in X \mid xRy\ \forall y \in X\}$: the set of most preferred alternatives with respect to R. The set $r_2(R)$ denotes the set $\{x \in X \setminus r_1(R) \mid xRy\ \forall y \in X \setminus r_1(R)\}$, and so on. When $r_k(R)$ consists of one alternative $x \in X$, x is *isolated* and for notational simplicity, I omit the braces and write $r_k(R) = x$ whenever the omission does not cause confusion.

Let d denote the *Kemeny distance* in $\mathcal{W}$ defined by for each $R_i, R_i' \in \mathcal{W}$, $d(R_i, R_i') = |R_i \Delta R_i'|$, the cardinality of the symmetric difference between R_i and R_i'. Note that the maximum distance in $\mathcal{W}$ is $m(m-1)$. For each $R_i \in \mathcal{W}$ and for each $k \in \{1, \ldots, m\}$, let $D(R_i, k)$ denote the set $\{R_i' \in \mathcal{W} \mid d(R_i, R_i') \leq k\}$; the set of the preference relations within k distance from R_i.

In the main theorem in Section 3, it will be seen that the crucial distance from the true preference relation is $(m-1)$. What makes this number $(m-1)$ is crucial is the fact that $D(R, m-1) \setminus \{R\} \neq \emptyset$ for all $R \in \mathcal{W}$.

An n-tuple $R_N = (R_1, \ldots, R_n) \in \mathcal{W}^N$ is a *preference profile*. Given a preference profile R_N, (R_i', R_{-i}) denotes the preference profile such that R_i is replaced by $R_i' \in \mathcal{W}$ in R_N.

[3] Every subset of $X \times X$ is *binary relation* on X. A binary relation R on X is *complete* if for any $x, y \in X$, either xRy or yRx holds, *transitive* if for any $x, y, z \in X$, $[xRy\ \&\ yRz]$ implies xRz, *antisymmetric* if for any $x, y \in X$, $[xRy\ \&\ yRx]$ implies $x = y$. A binary relation is a *weak order* if it is complete and transitive, a *linear order* if it is complete, transitive, and antisymmetric.

[4] A binary relation R on X is *reflexive* if for any $x \in X$, xRx holds, *symmetric* if for any $x, y \in X$, xRy implies yRx. A binary relation is an *equivalence relation* if it is complete, symmetric, and transitive.

A function f of $\mathcal{W}^N$ into X is called a *social choice function* or a *rule*.

For each $k \in \{1, \ldots, m(m-1)\}$, a rule satisfies $D(k)$*-proofness* if for every $R_N \in \mathcal{W}^N$ and for every $i \in N$,

$$f(R_N) R_i f(R'_i, R_{-i}), \quad \forall R'_i \in D(R_i, k).$$

Note that *strategy-proofness* and *AM-proofness* in Sato (2010b) are the two extremes in the sense that $D(m(m-1))$*-proofness* is *strategy-proofness* and $D(1)$*-proofness* is *AM-proofness*.

We say that $D(k)$*-proofness* is *(logically) equivalent* to *strategy-proofness* if every rule satisfies $D(k)$*-proofness* if and only if it satisfies *strategy-proofness*. (Because *strategy-proofness* always implies $D(k)$*-proofness* for any k, the problem of the equivalence is reduced to whether $D(k)$*-proofness* implies *strategy-proofness*.) When $D(k)$*-proofness* and *strategy-proofness* are equivalent, the restriction on the reportable false preferences assumed under $D(k)$*-proofness* does not any role to construct a nonmanipulable rule.

I state preliminary results.

Proposition 2.1

If $D(k)$-proofness is equivalent to strategy-proofness, then for every $k' \in \{k, k+1, \ldots, m(m-1)\}$, $D(k')$-proofness is equivalent to strategy-proofness.

Proof. Let k' be any element of $\{k, k+1, \ldots, m(m-1)\}$. Because it is clear that *strategy-proofness* implies $D(k')$*-proofness*, it suffices to show that $D(k')$*-proofness* implies *strategy-proofness*. Let f be any rule satisfying $D(k')$*-proofness*. Because $k' \geq k$, f satisfies $D(k)$*-proofness*. By the assumption that $D(k)$*-proofness* and *strategy-proofness* are equivalent, f satisfies *strategy-proofness*. Therefore, $D(k')$*-proofness* implies *strategy-proofness*. ■

Proposition 2.2 (Sato (2010b))

For every $k \leq m-2$, $D(k)$-proofness is not equivalent to strategy-proofness.

3 A main result

This section is devoted to the statement of a main theorem and its proof.

Theorem 3.1

$D(k)$-proofness is equivalent to strategy-proofness if and only if $k \geq m-1$.

Proof. By Proposition 2.2, the "only if" part follows. Moreover, when $D(m-1)$-*proofness* is equivalent to *strategy-proofness*, by Proposition 2.1, $D(k)$-*proofness* is equivalent to *strategy-proofness* for every $k \geq m-1$. Thus, it suffices to prove the equivalence of $D(m-1)$-*proofness* and *strategy-proofness*. Let f be any rule satisfying $D(m-1)$-*proofness*. Let R_N be any preference profile, let i be any agent, and let R'_i be any element of $D(R_i, m-1)$. Let $x = f(R_N)$, let $U(R_i) = \{y \in X \mid yP_ix\}$, and $L(R_i) = \{y \in X \mid xR_iy\}$. The set $U(R_i)$ is the (strict) upper contour set of x with respect to R_i and the set $L(R_i)$ is the lower contour set of x with respect to R_i. Out goal is to prove $f(R'_i, R_{-i}) \in L(R_i)$.

Let $\bar{R}_i$ be a linear order such that

(i) for every $y, z \in X$, yP_iz implies $y\bar{P}_iz$, and

(ii) for every $y \in X \setminus \{x\}$ with xI_iy, $x\bar{P}_iy$ holds.

In the linear order $\bar{R}_i$, the strict relations in R_i remain unchanged and the ties in the indifference class containing x is broken so that x is preferred to the other alternatives in the indifference class.

Lemma 3.1

$f(\bar{R}_i, R_{-i}) = x$ and the lower contour set of x with respect to $\bar{R}_i$, denoted by $L(\bar{R}_i)$, is $L(R_i)$.

Proof of Lemma 3.1. From R_i to $\bar{R}_i$, construct a sequence of weak orders $(R^1 \ldots R^\ell)$ from R_i to $\bar{R}_i$, i.e., $R^1 = R_i$ and $R^\ell = \bar{R}_i$ such that

Table 1: Preference relations in the proof of Lemma 3.1

R^1	R^2	R^3	R^4
xy	x	x	x
zwv	y	y	y
	zwv	z	z
		wv	w
			v

(i) for every $h \in \{1, \ldots, \ell - 1\}$, one indifference class is divided into consecutively ranked two indifference classes at least one of which is a singleton,

(ii) once an alternative y becomes isolated in R^h, y is isolated in $R^{h'}$ for all $h' \geq h$, and

(iii) x is isolated in R^2.

I explain a procedure of constructing such a sequence by an example. Let $X = \{x, y, z, w, v\}$ and R_i and $\bar{R}_i$ are R^1 and R^4 in Table 1, respectively. First, from R^1 to R^2, divide $r_1(R^1) = \{x, y\}$ into $r_1(R^2) = x$ and $r_2(R^2) = y$. Next, from R^2 to R^3, divide $r_3(R^2) = \{z, w, v\}$ into $r_3(R^3) = z$ and $r_4(R^3) = \{w, v\}$, and so on. This procedure can be easily generalized.

If we let $r_k(R^h)$ be the indifference class which is divided from R^h to R^{h+1}, $d(R^h, R^{h+1}) = |r_k(R^h)| - 1 \leq m - 1$ holds. Because f is $D(m-1)$*-proof,* $f(R^h, R_{-i}) R^h f(R^{h+1}, R_{-i})$ and $f(R^{h+1}, R_{-i}) R_{h+1} f(R^h, R_i)$ for all $h \in \{1, \ldots, \ell - 1\}$. Remember that $x = f(R^1, R_{-i})$ and x is isolated in R^2. This implies that $f(R^2, R_{-i})$ should be also x. Because x is isolated in R^h for all $h \geq 2$, the value of f is always x along the sequence $(R^1 \ldots R^\ell)$. Therefore, $f(R^\ell, R_{-i}) = f(\bar{R}_i, R_{-i}) = x$. Next, I prove $L(R_i) = L(\bar{R}_i)$. Because the alternatives ranked higher than x in $\bar{R}_i$ are precisely the ones ranked higher in R_i, the strict upper contour sets of x at R_i and $\bar{R}_i$ are the same. In other words, $L(R_i) = L(\bar{R}_i)$. ■

Let $\bar{R}'_i$ be a linear order such that

(i) for all y, z, yP'_iz implies $y\bar{P}'_iz$, and

(ii) for all $y, z \in X$ such that yI'_iz, $y \in U(R_i)$, and $z \in L(R_i)$, the relation $y\bar{P}'_iz$ holds.

The linear order $\bar{R}'_i$ can be obtained by breaking ties in R'_i partly based on R_i.

Lemma 3.2

$f(\bar{R}'_i, R_{-i}) \in L(R_i)$.

Proof of Lemma 3.2. By Lemma 3.1, it suffices to prove $f(\bar{R}'_i, R_{-i}) \in L(\bar{R}_i)$. In the proof of his main theorem, Sato (2010a) shows that there is a sequence of *linear* orders $(R^1 R^2 \dots R^\ell)$ from $\bar{R}_i$ to $\bar{R}'_i$ such that $f(R^h, R_{-i}) \in L(\bar{R}_i)$ implies $f(R^{h+1}, R_{-i})$ for all $h \in \{1, \dots, \ell - 1\}$. Therefore, it can be concluded that $f(\bar{R}'_i, R_{-i}) \in L(\bar{R}_i)$. ■

Let $a = f(R'_i, R_{-i})$. Suppose to the contrary that $a \in U(R_i)$.

From R'_i, construct a new linear order according to the following procedure:

Let $r_k(R'_i)$ be the indifference class containing a.

STEP 1: If $r_k(R'_i) = a$, then proceed to Step 3. If $r_k(R'_i)$ contains more at least two alternatives, raise a by one position.

STEP 2: Break ties in $r_k(R'_i) \setminus \{a\}$ based on $\bar{R}'_i$.

STEP 3: Break ties in the other indifference classes based on $\bar{R}'_i$.

Let $\tilde{R}'_i$ denote the resulting preference relation. By $D(m-1)$-*proofness*, $f(\tilde{R}'_i, R_{-i}) = a \in U(R_i)$.

Note that the only difference between $\bar{R}'_i$ and $\tilde{R}'_i$ is the position of a. More specifically, how the ties in $r_k(R'_i) \cap U(R'_i)$ are broken is the only difference between $\bar{R}'_i$ and $\tilde{R}'_i$.

Remember that $f(\bar{R}'_i, R_i) \in L(R_i)$. Then, raise a by one position at a time in $\bar{R}'_i$ until we have $\tilde{R}'_i$. However, the social outcome $f(\tilde{R}'_i, R_{-i})$ should be unchanged, i.e., $f(\tilde{R}'_i, R_{-i}) = f(\bar{R}'_i, R_i)$, which is a contradiction to $f(\tilde{R}'_i, R_{-i}) \in U(R_i)$. ■

4 Concluding remarks

This paper considers the universal domain of weak orders. In terms of the distance between reportable false preference relations and the sincere one, the range which is effective in constructing a nonmanipulable rule is characterized.

- If the distance belongs to $\{1, \dots, m-2\}$, then the agents' reluctance to make a large lie represented by the distance is effective, but
- if the distance belongs to $\{m-1, \dots, m(m-1)\}$, then the reluctance is ineffective.

This paper concentrates on nonmanipulability of a rule. However, usually, we want rules to satisfy additional properties such as unanimity. Study of the cases where rules satisfy additional properties is a future research.

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