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An essence of the impossibility for constructing strategy-proof social choice correspondences

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Abstract

This paper shows an essence of an impossibility for constructing strategy-proof social choice correspondences. I employ Benoit's (Journal of Economic Theory 102, 421–436, 2002) formulation of preferences over sets and prove that under each strategy-proof and unanimous social choice correspondence, there is at least one agent who is decisive. Although this is not a clear-cut impossibility result, this is nearly an impossibility for most purposes of social choice. Moreover, the existence of a decisive agent does not need the universal set of preferences over alternatives. Each circular set of preferences over alternatives is sufficient for the existence of a decisive agent.

Keywords: circular set, social choice correspondence, strategy-proofness

JEL classification: D71.

1 Introduction

The Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) shows that each *strategy-proof* and *unanimous* single-valued social choice rule is *dictatorial*. This paper addresses the following two questions:

(i) What can be said about *strategy-proof* and *unanimous* social choice correspondences, i.e., multi-valued social choice rules, with Benoît (2002)'s formulation of preferences over sets?

My answer is the existence of at least one agent who is "decisive". This is not a clear-cut impossibility result. However, for most purposes of social choice, this is nearly an impossibility result.

The other question is the following:

(ii) How robust the impossibility to restrictions on preferences over alternatives?

An answer is that the impossibility holds over each "circular set" of preferences over alternatives.

In the rest of this section, I discuss the questions and the answers in detail.

The literature on *strategy-proofness* shows that each *unanimous* and *nondictatorial* single-valued social choice rule violates *strategy-proofness* in most environments.[1] The literature on *strategy-proof social choice correspondences* investigates what can be done when we relax the assumption of single-valuedness. This paper assumes that a society is to declare the set of best alternatives.[2] The outcome can be a singleton, but if there are several indistinguishably best alternatives, the outcome are multi-valued.[3]

[1]We now have an excellent survey, Barberà (2010).

[2]Some other interpretations are compatible with the analysis of this paper. For example, each value of a social choice correspondence could be the result of a first screening process.

[3]Thus, for example, the problem of hiring exactly two new faculty members is out of the scope of this paper. See, for example, Özyurt and Sanver (2008) for an analysis of such a case.

In some other lines of research, an impossibility disappears by relaxing single-valuedness. For example, it is well known that *anonymity*, *neutrality*, and *Pareto efficiency*[4] are incompatible for single-valued social choice rules in some environments. (For example, three agents and m alternatives with $m \geq 3$.) However, by relaxing the single-valuedness assumption, the plurality rule, the Borda rule, and many other rules satisfy all of the axioms. We hope that this kind of escape from the impossibility occurs with *strategy-proof* social choice correspondences. However, it is seemingly safe to say that the literature confirms that such a escape is unachievable.[5] I scrutinize this assertion in this paper.

Benoît (2002) is one of the most important contributions in this topic. His theorem states that there is no *strategy-proof* and *nearly unanimous* social choice correspondence.[6] What makes his theorem significant is the set of intuitively clear, reasonable, and weak requirements on the structure of preferences over sets of alternatives. However, the desirability of *near unanimity* is somewhat controversial.[7] This paper clarifies the structure of *strategy-proof* social choice correspondences more completely with Benoît (2002)'s formulation of preferences over sets. I prove that under each *strategy-proof* and *unanimous* social choice correspondence, at least one agent is "decisive".[8] This theorem can give a new impossibility result as

[4] *Anonymity* requires a symmetric treatment of the agents, *neutrality* requires a symmetric treatment of the alternatives, *Pareto efficiency* requires that an alternative x does not belong to the social outcome if there exists an alternative y such that each agent prefers y to x.

[5] See Section 8 of Barberà (2010).

[6] *Near unanimity* is a property of a social choice correspondence. It requires that when all but one agent have a common most preferred alternative, then the alternative must be the social outcome at the preference profile.

[7] For example, the Borda rule violates *uear unanimity* in some cases such as three agents and three alternatives.

[8] I give the definition of decisiveness informally. Agent i is *decisive for an alternative* x if x belongs to the social outcome under a situation where there is an alternative y such that for agent i, x is the best alternative, and for the other agents, y is the best alternative and x is the worst alternative. Agent i is *decisive* if he is decisive for each alternative.

well as Benoît (2002)'s result as a corollary. Thus, Benoît (2002)'s formulation of *strategy-proofness* attains a broader implication.

These results are proved over each *circular set* of preferences over alternatives.[9] This is another contribution of this paper because the literature on *strategy-proof* social choice correspondences assumes the universal set of preferences over alternatives. There is a possibility that over some set of preferences over alternatives, there is no *strategy-proof* and "democratic" single-valued social choice rule while there is such a social choice correspondence. The problem of this possibility has been open. This paper shows that over each circular set of preferences over alternatives, there is no such possibility. In terms of cardinality, circular sets can be very small relative to the universal set of preferences. The minimal circular sets consist of $2m$ preference relations, where m is the number of alternatives. Thus, the robustness of the existence of a decisive agent is established.

The paper is organized as follows. Section 2 gives basic notation and definitions. Section 3 gives a main result and its applications. Section 4 concludes. Appendix contains a proof of the main result.

2 Basic notation and definitions

Let $N = \{1, \ldots, n\}$ be a finite set of *agents* with $n \geq 2$, and X be a finite set of *alternatives* with $|X| = m \geq 3$. Let $\mathcal{X}$ be the set of nonempty subsets of X. I write $x, y, z, ...$, for elements of X, and $A, B, C, ...$, for elements of $\mathcal{X}$. For each $x \in X$, I write x for $\{x\}$ when the omission of the braces does not cause confusion. Let L be the set of linear orders on X, and $\mathcal{L}$ be the set of weak orders on $\mathcal{X}$.[10] Typical

[9] The concept of circular sets is introduced by Sato (2010). Sato (2010) proves that on each circular domain, each *strategy-proof* and *unanimous* single-valued social choice rule is *dictatorial*.

[10] Let S be a set. Each subset of $S \times S$ is a *binary relation* on S. A binary relation R on S is *complete* if for each pair $x, y \in S$, either xRy or yRx, *transitive* if for each triple $x, y, z \in S$, $[xRy$ & $yRz]$ implies xRz, *antisymmetric* if for each pair $x, y \in S$, $[xRy$ & $yRx]$ implies $x = y$.

notation for an element of L is R, and when the preference relation belongs to a particular agent $i \in N$, then we write it as R_i. Typical notation for an element of $\mathcal{L}$ is $\mathcal{R}$, and when it belongs to a particular agent $i \in N$, then we write it as $\mathcal{R}_i$. The strict relations of R and $\mathcal{R}$ are P and $\mathcal{P}$, respectively. For each $R \in L$ and each $\mathcal{R} \in \mathcal{L}$, let $r_k(R)$ and $r_k(\mathcal{R})$ denote the kth ranked elements of X and $\mathcal{X}$, respectively. In defining some kinds of preferences over sets, the best and the worst alternatives of each $A \in \mathcal{X}$ play an important role. In such a case, let $\max(R, A)$ and $\min(R, A)$ denote the best and the worst alternatives of $A \in \mathcal{X}$ according to $R \in L$, respectively.

For each $\mathcal{D} \subset \mathcal{L}$, a *social choice correspondence* (or simply a *rule*) on $\mathcal{D}$ is a function from $\mathcal{D}^N$ into $\mathcal{X}$. Thus, each rule maps each preference profile over sets to a set.

For each $D \subset L$, a correspondence E from D into $\mathcal{L}$ is an *extension rule* if it satisfies the properties **A1** through **A4** listed below. For each $R \in \mathcal{D}$, $E(R)$ is interpreted as the set of *plausible preferences* on $\mathcal{X}$ with respect to R. I require an extension rule to satisfy the following axioms:

A1 For each $R \in D$, each $\mathcal{R} \in E(R)$, and each $A \neq r_1(R)$, we have $r_1(R)\mathcal{P}A$.

A2 For each $R \in D$, each $\mathcal{R} \in E(R)$, and each $A \notin \{r_1(R), \{r_1(R), r_2(R)\}\}$, we have $\{r_1(R), r_2(R)\}\mathcal{P}A$.

A3 For each $R \in D$, each $\mathcal{R} \in E(R)$, and each $A \neq r_m(R)$, we have $A\mathcal{P}r_m(R)$.

Moreover, I assume the following axiom:

A4 (The existence of an r_2-favoring preference relation) For each $R \in D$, there exists $\mathcal{R} \in E(R)$ such that for each $A \notin \{r_1(R), \{r_1(R), r_2(R)\}, r_2(R)\}$, we have $r_2(R)\mathcal{P}A$.

A binary relation is a *weak order* if it is complete and transitive, a *linear order* if it is complete, transitive, and antisymmetric.

These axioms are based on Benoît (2002). **A1** says that the singleton consisting of the top ranked alternative is always preferred to the other sets of alternatives. **A2** says that the set consisting of the top and the second ranked alternatives is always preferred to the other sets of alternatives except the singleton $r_1(R)$. **A3** says that the worst alternative is always less preferred to the other sets of alternatives. **A4** is different from the earlier axioms: it does not say about each preference relation over sets. It requires the *existence* of a preference relation over sets such that the singleton $r_2(R)$ is preferred to the sets of alternatives except $r_1(R)$ and $\{r_1(R), r_2(R)\}$. Such a preference relation is called an r_2*-favoring preference relation*.

These axioms are equivalently stated as follows: each $\mathcal{R} \in E(R)$ and some $\mathcal{R}' \in E(R)$ have the following structure for each $R \in D$:

	$\mathcal{R}$	$\mathcal{R}'$
top	$r_1(R)$	$r_1(R)$
second	$\{r_1(R), r_2(R)\}$	$\{r_1(R), r_2(R)\}$
third	$\vdots$	$r_2(R)$
$\vdots$	$\vdots$	$\vdots$
bottom	$r_m(R)$	$r_m(R)$

In the above table, $\mathcal{R}'$ is r_2-favoring.

In the following, I give several examples of extension rules.[11]

Example 2.1

For each $R \in D$, let $E^{min}(R)$ be the leximin relation. The leximin relation is defined as follows. Let $\mathcal{R} = E^{min}(R)$. Let $A, B \in \mathcal{X}$. First, compare the worst alternatives of A and B according to R. If $\min(R, A) P \min(R, B)$, then A is preferred to B, i.e., $A\mathcal{P}B$. If $\min(R, B) P \min(R, A)$, then $B\mathcal{P}A$. If the worst

[11]Example 2.1 is the same as Example 3 by Benoît (2002). Example 2.2 is a special case of Example 2 by Benoît (2002).

alternatives are idential, then compare the second worst alternatives of A and B according to R. By continuing this way, a preference over A and B is determined. If one of A and B, say A, is exhausted before a preference over A and B is determined, then B is preferred to A. It can be seen that E^{min} is an extension rule.

Example 2.2

Let $R \in D$. For each $k \in \{1, \ldots, m\}$, let

$$u(r_k(R)) = \frac{m(m-1)}{2} - 1 - \cdots - (k-1) - k = \frac{m(m-1)}{2} - \frac{k(k-1)}{2}.$$

When $m = 10$, 45 is assigned to $r_1(R)$, 44 to $r_2(R)$, 42 to $r_3(R)$, ..., 0 to $r_m(R)$. Define $E^{exp}(R)$ be the singleton $\mathcal{R}$ defined by for each pair $A, B \in \mathcal{X}$,

$$A\mathcal{R}B \iff \sum_{a \in A} \frac{u(a)}{|A|} \geq \sum_{b \in B} \frac{u(b)}{|B|}.$$

Then, E^{exp} is also an extension rule.

Example 2.3

For each $R \in \mathcal{L}$, let $E^{wb}(R)$ be the singleton $\mathcal{R}$ defined by for each pair $A, B \in \mathcal{X}$,

$$A\mathcal{R}B \iff \begin{bmatrix} \min(R,A) R \min(R,B) \\ \text{or} \\ \begin{bmatrix} \min(R,A) = \min(R,B) \\ \text{and} \\ \max(R,A) R \max(R,B) \end{bmatrix} \end{bmatrix}$$

The preference relation $\mathcal{R}$ compares A and B by their worst and the best alternatives. The primary criterion is the worst alternative and the secondary criterion is the best alternative. If the worst alterantive of A is preferred to the worst alternative of B according to R, then A is preferred to B. If the worst alternatives of A and B are the same, then compare the best alternatives of A and B. If the best alternative of A is preferred to that of B, then A is preferred to B. E^{wb} defined in this way is an extension rule.[12]

[12]The superscript "wb" stands for "Worst and Best".

For each $D \subset L$ and each extension rule E, let $\mathcal{D}(D, E) = \bigcup_{R \in D} E(R)$. This is the set of plausible preferences over $\mathcal{X}$ if preferences over X belong to D and they are extended over $\mathcal{X}$ through E.

Definition 2.1

For each $D \subset L$ and each extension rule E, a rule f on $\mathcal{D}(D, E)$ is

- *strategy-proof* if for each $\mathcal{R}_N \in \mathcal{D}(D, E)^N$, each $i \in N$, and each $\mathcal{R}'_i \in \mathcal{D}(D, E)$, we have $f(\mathcal{R}_N)\mathcal{R}_i f(\mathcal{R}'_i, \mathcal{R}_{-i})$, and
- *unanimous* if for each $x \in X$ and each $\mathcal{R}_N \in \mathcal{D}(D, E)^N$ such that $r_1(\mathcal{R}_i) = x$ for each $i \in N$, we have $f(\mathcal{R}_N) = x$.

When a rule is *strategy-proof*, reporting his true preference relation is always an optimal strategy regardless of what the other agents report. In other words, each agent does not have an incentive to lie. When a rule is *unanimous*, complete agreement on the best alternative is respected.

Definition 2.2

For each rule f on $\mathcal{D}(D, E)$,

- agent $i \in N$ is *decisive for* $x \in X$ if there exists $y \in X \setminus \{x\}$ such that for each $\mathcal{R}_N \in \mathcal{D}(D, E)^N$ having the structure

i	$\forall j \in N \setminus \{i\}$		
x	y	$\dots$	y
$\vdots$	$\vdots$	$\vdots$	$\vdots$
	x	$\dots$	x

  where $\mathcal{R}_i$ is r_2-favoring, we have $x \in f(\mathcal{R}_N)$, and
- agent i is *decisive* if he is decisive for each $x \in X$.

When an agent is decisive for $x \in X$, then he has a power to have the outcome contain x even if the other agents collusively report that $y \in X \setminus \{x\}$ is the best and x is the worst alternative. The existence of a decisive agent does not necessarily mean an "undemocratic" rule. For example, the rule f defined by $f(\mathcal{R}_N) = \bigcup_{i \in N} r_1(\mathcal{R}_i)$ is *anonymous* and each agent is decisive. However, most, if not all, selective and "democratic" rules[13] do not admit the existence of a decisive agent.[14] Therefore, the existence of an decisive agent is nearly an impossibility result for most purposes of social choice.

Next, I define a class of sets of preferences over alternatives.

Definition 2.3

$D \subset L$ is *circular* if the elements of X can be indexed $x_1, x_2, \ldots, x_m$ so that for each $x_k \in \{x_1, \ldots, x_m\}$,

(i) there exists $R \in D$ such that $r_1(R) = x_k$, $r_2(R) = x_{k+1}$, $r_m(R) = x_{k-1}$,

(ii) there exists $R' \in D$ such that $r_1(R') = x_k$, $r_2(R') = x_{k-1}$, $r_m(R') = x_{k+1}$.

(Let $x_0 = x_m$ and $x_{m+1} = x_1$.)

When D is a circular set, there exists a way of numbering the alternatives such that for each x_k, there exist $R, R' \in D$ satisfying the two conditions. The two conditions put restrictions on the positions of x_k and its neighbors x_{k-1} and x_{k+1}. In both R and R', x_k is top ranked, but the positions of x_{k-1} and x_{k+1} are interchanged.

On each circular set of preferences, each *strategy-proof* and *unanimous* single-valued social choice rule is *dictatorial* (Sato, 2010). Therefore, with single-valued social choice rules, the impossibility holds on each circular set of preferences.

3 Results

This section presents a main result and its applications.

[13]For example, the plurality rule, the Borda rule, each Condorcet consistent rule, and so on.

[14]This argument assumes $n \geq 3$.

Theorem 3.1

Let D be circular, and f be a unanimous *and* strategy-proof *rule on $\mathcal{D}(D, E)$. Then, there exists $i \in N$ who is decisive.*

This is not like a typical impossibility result, but it uncovers an important feature of each *unanimous* and *strategy-proof* rule. We know that with additional axioms on rules or additional requirements on the structure of preferences over sets, we have clear-cut impossibility results. (For example, Barberà, Dutta, and Sen (2001); Benoît (2002); Özyurt and Sanver (2009).) My main result is not such a clear-cut impossibility result. It finds that a set of intuitively plausible and weak requirements on preferences over sets leads to the existence of a decisive agent under each *strategy-proof* and *unanimous* rule. The interpretation of this result depends on the context. If the only purpose of the rule designer is to construct a *strategy-proof*, *unanimous*, and "democratic" rule, then its purpose might be accomplished.[15] However, if the rule designer wants to avoid the existence of a decisive agent (this would be the case in most cases), we have the impossibility.

Moreover, the theorem is proved with each circular set of preferences over alternatives. This shows the robustness of the result to reduction of the number of preference relations over alternatives.

I give several examples of *strategy-proof* and *unanimous* rules. By Theorem 3.1, there should be at least one decisive agent.

Example 3.1

Consider the extension rule E^{wb} introduced in Example 2.3. On $\mathcal{D}(D, E^{wb})$ where D is circular, the Pareto rule choosing the set of *Pareto efficient* alternatives is *strategy-proof* and *unanimous*. This statement can be established by Feldman (1979)'s proofs of his Theorems 3 and 4. In this case, each agent is decisive.

[15]For example, when the agents have the leximin preferences over sets, the rule f defined by $f(\mathcal{R}_N) = \bigcup_{i \in N} r_1(\mathcal{R}_i)$ is *strategy-proof* and *unanimous*. This fact is pointed out by Özyurt and Sanver (2009).

Example 3.2

Consider the extension rule E^{wb} again. On $\mathcal{D}(D, E^{wb})$ where D is circular, define the rule f by for each $\mathcal{R}_N \in \mathcal{D}(D, E^{wb})^N$,

$$f(\mathcal{R}_N) = \{x \in X \mid x\mathcal{R}_1 r_1(\mathcal{R}_2) \text{ and } x\mathcal{R}_2 r_1(\mathcal{R}_1)\}.$$

I explain this rule. Let $\mathcal{R}_N$ be a preference profile. Let A be the set of alternatives which are "between" $r_1(\mathcal{R}_1)$ and $r_1(\mathcal{R}_2)$ in $\mathcal{R}_1$ (including $r_1(\mathcal{R}_1)$ and $r_1(\mathcal{R}_2)$). Similarly, let B be the set of alternative which are "between" $r_1(\mathcal{R}_2)$ and $r_1(\mathcal{R}_1)$ in $\mathcal{R}_2$. The outcome is the intersection of these two sets A and B. For example, when $\mathcal{R}_N$ has the structure

1	2	others
x	w	$\vdots$
y	y	
z	z	
w	x	
$\vdots$	$\vdots$	

where $x, y, z, w \in X$, then $A = B = \{x, y, z, w\}$ and the social outcome is $\{x, y, z, w\}$. This rule is *strategy-proof* and *unanimous*, and agents 1 and 2 are decisive. A notable feature of this rule is that alternatives contained in the outcome are *not* necessarily *Pareto efficient* as evidenced by the above arguments. (The alternative z is dominated by y.)

These examples show that the scope of Theorem 3.1 is quite wide. Moreover, from the theorem, new impossibility results readily follow.

An alternative x is the *Condorcet loser* at $\mathcal{R}_N \in \mathcal{L}^N$ if for each $y \in X \setminus \{x\}$, a strict majority of the agents prefers y to x. The Condorcet loser does not always exist. If it exists, excluding the Condorcet loser from the social outcome is a reasonable requirement.

Corollary 3.1

Let D be circular and $n \geq 3$. Then, there is no unanimous *and* strategy-proof *rule f on $\mathcal{D}(D, E)$ whose outcome never contains the Condorcet loser.*

Actually, I can state a stronger result. Under the assumption of Corollary 3.1, let f be a *unanimous* and *strategy-proof* rule. Then, for each $x \in X$, there exists $\mathcal{R}^N \in \mathcal{D}(D, E)^N$ at which x is the Condorcet loser and $x \in f(\mathcal{R}^N)$.

A rule f on $\mathcal{D}(D, E)$ is *nearly unanimous* if for each $x \in X$ and each $\mathcal{R}_N \in \mathcal{D}(D, E)^N$ such that $r_1(\mathcal{R}_i) = x$ for at least $n - 1$ agents, we have $f(\mathcal{R}_N) = x$. The existence of an agent who is decisive means the violation of *near unanimity*. Thus, we have Benoît (2002)'s impossibility theorem as a corollary. Precisely speaking, the following statement is stronger than Benoît (2002) because the impossibility is stated over each circular set of preferences over alternatives.

Corollary 3.2

Let D be circular and $n \geq 3$. Then, there is no strategy-proof *and* nearly unanimous *rule f on $\mathcal{D}(D, E)$.*

4 Concluding remarks

Recall that the requirement on the structure of preferences over sets is so weak that most cases fall within the scope of this paper, and that my result, Theorem 3.1, does not rely on any additional properties of rules. Moreover, the result holds on each circular set of preferences over alternatives, and the class of circular sets encompasses many situations. In sum, my theorem is almost always relevant. Also, the existence of a decisive agent is not acceptable when the society is to delcare the set of best alternatives. In this sense, this paper gives an essence of the impossibility for constructing *strategy-proof* multi-valued social choice rules.

However, the scope of this paper is not unlimited. First, some classes of preferences over sets are outside the scope of this paper. (For example, the domain consisting of the "leximax" preferences[16] over sets does not contain an r_2-favoring preference relation.) Second, I am not sure whether the same results hold with other sets of preferences over alternatives such as "linked" ones (Aswal, Chatterji, and Sen, 2003).

Finally, I mention two approaches in this topic, i.e., *strategy-proofness* of multi-valued social choice rules. One approach is considering a function which maps each preference profile over *sets* to a set. The other one is considering a correspondence which maps each preference profile over *alternatives* to a set.[17] I employ the former one just because it is more general than the latter one. However, the set of all nonempty subsets of X might be very large. From a practical point of view, in such a case, it might be natural to restrict our attention to rules which depend on information on preferences over *alternatives*. Then, the latter approach is sufficient.

Appendix

A proof of Theorem 3.1

The idea of a main part of the proof is the combination of that of Benoît (2002) and Sato (2010).[18]

Assign a number from 1 to m to each alternative so that it makes D circular. Let $x_1, \ldots, x_m$ be the indexed alternatives. Let $x_k \in \{x_1, \ldots, x_m\}$ and I prove that there exists an agent who is decisive for x_k. Unless otherwise stated, each preference relation over sets is r_2-favoring.

STEP 1: Consider $\mathcal{R}_N^1 \in \mathcal{D}(D, E)^N$ having the structure in the following table:

[16] Leximax preferences are the counterpart of leximin preferences.

[17] This approach first appears in Barberà, Dutta, and Sen (2001).

[18] These works also benefit from earlier ones. For example, Benoît (2002) benefits from Barberà (1983), and Sato (2010) benefits from Sen (2001).

$\mathcal{R}^1_N$			
1	…	n	Outcome
x_{k+1}	…	x_{k+1}	x_{k+1}
x_{k+2}	…	x_{k+2}	
⋮	⋮	⋮	
x_k	…	x_k	

Unanimity implies $f(\mathcal{R}^1_N) = x_{k+1}$. (The last column of the above table showing the outcome is a logical consequence. It is not a constraint on the choice of $\mathcal{R}^1_N$.)

STEP 2: I find a *pivotal* agent i^* by the following way. For each $i \in N$, let $R^2_i \in D$ such that $r_1(R^2_i) = x_k$, $r_2(R^2_i) = x_{k+1}$, and $r_m(R^2_i) = x_{k-1}$. Let $\mathcal{R}^2_i \in E(R^2_i)$. Consider the following successive change of preferences. The starting point is $\mathcal{R}^1_N$. First, agent 1, next, agent 2, . . . , and finally, agent n change their preferences from $\mathcal{R}^1_i$ to $\mathcal{R}^2_i$. By unanimity, there is $i^* \in N$ such that

- after agent $i^* - 1$ changes his preference, the social outcome is not x_k:

$\left(\mathcal{R}^2_{\{1,\dots,i^*-1\}}, \mathcal{R}^1_{\{i^*,\dots,n\}}\right)$							
1	…	$i^* - 1$	i^*	$i^* + 1$	…	n	Outcome
x_k	…	x_k	x_{k+1}	x_{k+1}	…	x_{k+1}	x_{k+1}
x_{k+1}	…	x_{k+1}	x_{k+2}	x_{k+2}	…	x_{k+2}	or
⋮	⋮	⋮	⋮	⋮	⋮	⋮	$\{x_k, x_{k+1}\}$
x_{k-1}	…	x_{k-1}	x_k	x_k	…	x_k	

  (By *strategy-proofness*, the social outcome is either x_{k+1} or $\{x_k, x_{k+1}\}$.)

- After agent i^* changes his preference, the social outcome is x_k:

$\left(\mathcal{R}^2_{\{1,\ldots,i^*\}}, \mathcal{R}^1_{\{i^*+1,\ldots,n\}}\right)$							
1	$\ldots$	i^*-1	i^*	i^*+1	$\ldots$	n	Outcome
x_k	$\ldots$	x_k	x_k	x_{k+1}	$\ldots$	x_{k+1}	x_k
x_{k+1}	$\ldots$	x_{k+1}	x_{k+1}	x_{k+2}	$\ldots$	x_{k+2}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{k-1}	$\ldots$	x_{k-1}	x_{k-1}	x_k	$\ldots$	x_k	

STEP 3: Consider the first preference profile in Step 2. When agent i^* changes his preference relation so that we have the following profile, by *strategy-proofness*, the social outcome does not change.

1	$\ldots$	i^*-1	i^*	i^*+1	$\ldots$	n	Outcome
x_k	$\ldots$	x_k	x_{k+1}	x_{k+1}	$\ldots$	x_{k+1}	x_{k+1}
x_{k+1}	$\ldots$	x_{k+1}	x_k	x_{k+2}	$\ldots$	x_{k+2}	or
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\{x_k, x_{k+1}\}$
x_{k-1}	$\ldots$	x_{k-1}	x_{k+2}	x_k	$\ldots$	x_k	

STEP 4: In the following steps, I prove that i^* is decisive for x_k. Consider $\mathcal{R}^3_N \in \mathcal{D}(D,E)^N$ having the following structure, where for each $j \in N \setminus \{i^*\}$, $\mathcal{R}^3_j$ is not necessarily r_2-favoring. ($\mathcal{R}^3_{i^*}$ is r_2-favoring.)

$\mathcal{R}^3_N$				
i^*	$\forall j \in N \setminus \{i^*\}$			Outcome
x_k	x_{k-1}	$\ldots$	x_{k-1}	x_{k-1}
x_{k-1}	x_{k-2}	$\ldots$	x_{k-2}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{k+1}	x_k	$\ldots$	x_k	

I claim $x_k \in f(\mathcal{R}^3_N)$, i.e., agent i^* is decisive for x_k. Suppose not. Then, the social outcome should be x_{k-1}. If not, agent i^* benefits from misreporting that x_{k-1} is the top ranked alternative.

From $\mathcal{R}_N^3$, consider the successive change of preferences of the agents in $N \setminus \{i^*\}$ to the following preference profile $\mathcal{R}_N^4$: (In this profile, for each $j \in N \setminus \{i^*\}$, $\mathcal{R}_j^4$ is r_2-favoring.)

$\mathcal{R}_N^4$				
i^*	$\forall j \in N \setminus \{i^*\}$			Social outcome
x_k	x_{k-1}	$\dots$	x_{k-1}	x_{k-1}
x_{k-1}	x_k	$\dots$	x_k	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{k+1}	x_{k-2}	$\dots$	x_{k-2}	

By *strategy-proofness,* the social outcome is x_{k-1}.

Finally, agent i^* changes his preferences from $\mathcal{R}_{i^*}^4$ to the one in the following preference profile $\mathcal{R}_N^5$:

$\mathcal{R}_N^5$				
i^*	$\forall j \in N \setminus \{i^*\}$			Social outcome
x_k	x_{k-1}	$\dots$	x_{k-1}	x_{k-1}
x_{k+1}	x_k	$\dots$	x_k	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{k-1}	x_{k-2}	$\dots$	x_{k-2}	

By *strategy-proofness*, $f(\mathcal{R}_N^5) \notin \{\{x_{k-1}, x_k\}, x_k\}$. Then, the social outcome should be x_{k-1}. (Suppose not. When the agents in $N \setminus \{i^*\}$ successively misreport that x_k is the top ranked alternative, the social outcome becomes x_k at some stage. This is a beneficial change for the agent whose change makes the social outcome x_k. This is a contradiction to *strategy-proofness*.)

Strategy-proofness and $f(\mathcal{R}_N^5) = x_{k-1}$ together imply that as long as the agents in $N \setminus \{i^*\}$ rank x_{k-1} at the top, the social outcome is x_{k-1}.

STEP 5: Consider the following $\mathcal{R}_N^6$:

$\mathcal{R}^6_N$							
1	...	i^*-1	i^*	i^*+1	...	n	Outcome
x_{k-1}	...	x_{k-1}	x_{k+1}	x_{k-1}	...	x_{k-1}	x_{k-1}
x_k	...	x_k	x_k	x_{k-2}	...	x_{k-2}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{k-2}	...	x_{k-2}	x_{k+2}	x_k	...	x_k	

By Step 4, the social outcome at $\mathcal{R}^6_N$ is x_{k-1}. Consider that the agents in $\{1, \ldots, i^*-1\}$ successively change their preferences from $\mathcal{R}^6_i$ to $\mathcal{R}^7_i$:

$\mathcal{R}^7_N$							
1	...	i^*-1	i^*	i^*+1	...	n	Outcome
x_k	...	x_k	x_{k+1}	x_{k-1}	...	x_{k-1}	
x_{k-1}	...	x_{k-1}	x_k	x_{k-2}	...	x_{k-2}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{k+1}	...	x_{k+1}	x_{k+2}	x_k	...	x_k	

By *strategy-proofness*, the candidates for the social outcome are x_{k-1}, $\{x_{k-1}, x_k\}$, and x_k. If the outcome is either x_{k-1} or $\{x_{k-1}, x_k\}$, then agent i^* benefits from misreporting $\mathcal{R}^2_{i^*}$. This is a contradiction to *strategy-proofness*. Assume that the outcome is x_k. Then, after the agents in $N \setminus \{i^*\}$ changes their preferences to the ones in Step 3, the social outcome is still x_k. This is a contradiction to what we have in Step 3.

Therefore, we can conclude that agent i^* is decisive for x_k.

STEP 6: I prove that agent i^* is decisive for x_{k-1}. Consider $\mathcal{R}^8_N$ such that

$\mathcal{R}^8_N$			
1	$\dots$	n	Outcome
x_k	$\dots$	x_k	x_k
x_{k+1}	$\dots$	x_{k+1}	
$\vdots$	$\vdots$	$\vdots$	
x_{k-1}	$\dots$	x_{k-1}	

Unanimity implies $f(\mathcal{R}^8_N) = x_k$. Consider the successive change of preferences of the agents in $N \setminus \{i^*\}$ so that we have

$\mathcal{R}^9_N$				
i^*	$\forall j \in N \setminus \{i^*\}$			Social outcome
x_k	x_{k-1}	$\dots$	x_{k-1}	x_k
x_{k+1}	x_k	$\dots$	x_k	or
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\{x_{k-1}, x_k\}$
x_{k-1}	x_{k-2}	$\dots$	x_{k-2}	

The social outcome at $\mathcal{R}^9_N$ is not x_{k-1}. If it is, by *strategy-proofness*, $f(\mathcal{R}^3_N) = x_{k-1}$. This is a contradiction to what we proved. (Remember that $x_k \in f(\mathcal{R}^3_N)$.) *Strategy-proofness* implies that the social outcome is either x_k or $\{x_{k-1}, x_k\}$. By *unanimity*, when agent i^* changes his preference relation to the same one as the other agents, the social outcome becomes x_{k-1}. Thus, agent i^* is again "pivotal". Then, by repeating the arguments in Steps 3 through 5, it can be proved that agent i^* is decisive for x_{k-1}.

STEP 7: Now, we can apply mathematical induction. Steps 1 through 5 prove that agent i^* is decisive for x_k. This is an induction base. Step 6 proves that if agent i^* is decisive for x_k, then he is also decisive for x_{k-1}. This is an induction step. Therefore, by induction on the indices of the alternatives, we can conclude that agent i^* is decisive for each alternative, i.e., he is decisive. ■

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