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Abstract

A social welfare function satisfies *smooth response* if the smallest change in the variable (i.e., preference profile) leads to the smallest change, if any, in the value (i.e., social preference). This paper shows that each social welfare function on each *connected* domain satisfies *smooth response* and *nonmanipulability* if and only if it satisfies *monotonicity* and *independence of irrelevant alternatives (IIA)*. Thus, with additional properties, the equivalence of two major properties, *nonmanipulability* and *IIA*, is established.

Keywords: independence of irrelevant alternatives, nonmanipulability, smooth response, social welfare function

JEL classification: D71.

1 Introduction

Understanding the possibility of constructing a "nice" social welfare function is the starting point of a modern social choice theory (Arrow, 1951, 1963). This paper focuses mainly on three properties of social welfare functions, *smooth response*, *nonmanipulability*, and *independence of irrelevant alternatives (IIA)*. Before I explain my result, I briefly discuss *smooth response* and *nonmanipulability*.

In the same spirit of continuity,[1] a social welfare function is said to satisfy *smooth response* if the smallest change in the variable (i.e., preference profile) leads to the smallest change, if any, in the value (i.e., social preference). *Smooth response* is not a normative axiom, but it is so natural that we usually expect social welfare function to satisfy. You would be surprised if your tiny change of preferences leads to a substantial change of the social preferences.

I also introduce a new concept of *nonmanipulability* of social welfare functions. To define *nonmanipulability* of social welfare functions, we have to formulate agents' preferences over the social preferences. In this paper, we adopt the idea that R' *is more accordant with* R_i *than* R if

(i) for each alternatives x and y, if the preferences over x and y according to R_i and R are the same, then those according to R_i and R' are the same, and

(ii) for some alternatives x and y, the preferences over x and y according to R_i and R disagree, while those according to R_i and R' are the same.

Assume that R is the social preference under the sincere preference R_i of agent i, and that R' is the social preference under a false preference R'_i. The agent has an incentive to misreport R'_i if R' is more accordant with R_i than R. In this case, the social welfare function is *manipulable*, and it is *nonmanipulable* if it is not

[1] In an introductory course on real analysis, a function is continuous if "a small change in the variable leads to a small change in the value."

manipulable. If we restrict the options of misrepresentation to the preferences adjacent to the sincere one, we have *AP-nonmanipulability*, where "AP" stands for "Adjacent Preferences".

A main result of this paper is that on each *connected* domain, the following statements are equivalent:

- Each social welfare function satisfies *smooth response* and *nonmanipulability*
- Each social welfare function satisfies *monotonicity* and *independence of irrelevant alternatives (IIA)*.
- Each derived social choice function from each social welfare function is *strategy-proof*.

From this result, we have at least five important implications.

First, we find a meaning of *IIA* in terms of *smooth response* and *nonmanipulability* of social welfare functions. Understanding each axioms imposed on social welfare functions by Arrow (1951, 1963) is important to understand the implication of the theorem. Among the axioms, *IIA* would be the most controversial one. We show that under *monotonicity*, *IIA* can be split into *smooth response* and *nonmanipulability*.

The second implication of my result is about *nonmanipulability* of social welfare functions. Although *nonmanipulability* of social choice rules (such as social choice functions and social choice correspondences) is one of the central topics in social choice theory[2], *nonmanipulability* of social welfare functions has not been subject to close scrutiny. The only exception is Bossert and Storcken (1992) as I will discuss shortly. We show that we can use existing results on social welfare

[2]See Barberà (2010) for a survey. The Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) is the seminal result.

functions satisfying *IIA* to see the possibility of constructing a *nonmanipulable* social welfare function satisfying *smooth response*. For example, we have an impossibility theorem on the universal domain, whereas we have a possibility theorem on a set of single-peaked preferences.

The third implication is the significance of *connected* domains. A domain is *connected* if for each pair of preferences, a "path" between them exists in the domain. Our theorem and its proofs demonstrate that *connected* domains lead to the equivalence of properties. The equivalence of *nonmanipulability* and *AP-nonmanipulability*, *IIA* and *A-IIA*,[3] and *monotonicity* and *BM-monotonicity*[4] under *IIA*.

The fourth implication is a meaning of *strategy-proofness* of each derived social choice function. Our theorem shows that *strategy-proofness* of each derived social choice function is equivalent to *smooth response* and *nonmanipulability* of social welfare functions.

The fifth implication is about *smooth response*. This new property is the key in our theorem. Imposing *smooth response* is vanishing the difference of distinct notions. For example, as I mentioned above, under *smooth response*, *nonmanipulability* of social welfare functions and *strategy-proofness* of each social choice function are equivalent.

Finally, I mention closely related papers, Blair and Muller (1983) and Bossert and Storcken (1992). Blair and Muller (1983) show that each social welfare function satisfies *IIA* and *BM-monotonicity*[5] if and only if each derived social choice function is *strategy-proof*.[6] Bossert and Storcken (1992)'s concept of *nonmanipu-*

[3] *A-IIA* is a weaker condition than *IIA*.

[4] *BM-monotonicity* is used by Blair and Muller (1983). It is stronger than *monotonicity*.

[5] In their term, *monotonicity*.

[6] Blair and Muller (1983)'s theorem involves other properties as well. I borrow this simpler version from Moulin (1988).

lability of social welfare function is based on the Kemeny distance[7] between individual and social preferences. Bossert and Storcken (1992) establish that there is no *nonmanipulable*[8] social welfare function satisfying additional properties.[9]

The paper is organized as follows. Section 2 gives basic notation and definitions. Section 3 presents a main theorem and its proof. Section 4 concludes.

2 Basic notation and definitions

Let $N = \{1, \ldots, n\}$ be a set of *agents*, and X be a set of *alternatives*. Let $m = |X|$. Let $\mathcal{L}$ be the set of all *linear orders* on X.[10] A member of $\mathcal{L}$ is called a *preference relation*. We write R_i for agent i's preference relation. The strict relation of R_i is P_i.[11] An n-tuple $\boldsymbol{R} = (R_1, \ldots, R_n) \in \mathcal{L}^N$ is a *preference profile*. When R_i in $\boldsymbol{R}$ is replaced by R'_i, we write $(R'_i, \boldsymbol{R}_{-i})$ for the new preference profile. For each $A \subset X$, let $R_i|A$ be the restriction of $R_i \in \mathcal{L}$ to A. Let $\boldsymbol{R}|A = (R_1|A, \ldots, R_n|A)$. For each $\mathcal{D} \subset \mathcal{L}$, a function f from $\mathcal{D}^N$ into $\mathcal{L}$ is a *social welfare function on* $\mathcal{D}$, and $\mathcal{D}$ is the *domain* of f. f and $\mathcal{D}$ are our generic notation for a social welfare function and its domain.

For each $R_i \in \mathcal{L}$, $r_k(R_i)$ is the kth ranked alternative according to R_i. For each $A \subset X$, $r_k(R_i, A)$ is the kth ranked alternative according to R_i in A. Alternatives x and y are *adjacent in* R_i if they are consecutively ranked in R_i, i.e., $\{r_k(R_i), r_{k+1}(R_i)\} = \{x, y\}$ for some $k \in \{1, \ldots, m-1\}$. For each adjacent alternatives x, y in R_i, let $R_i^{x,y} \in \mathcal{L}$ be such that the only difference between $R_i^{x,y}$ and R_i is $xR_i^{x,y}y \iff yR_ix$. In R_i, by exchanging the positions of adjacent

[7] See Kemeny (1959) and Kemeny and Snell (1962).

[8] Bossert and Storcken (1992) adopt *coalitional strategy-proofness*.

[9] Bossert and Storcken (1992) assume that the number of alternatives is at least 4. Additional properties are *nonimposition* and *weak extrema independence* for even number of agents, and *nonimposition* and *extrema independence* for any number of agents.

[10] A binary relation is called a linear order if it is complete, transitive, and antisymmetric.

[11] $P_i = R_i \setminus \{(x, x) \mid x \in X\}$.

alternatives x and y, we have $R_i^{x,y}$.

For each $R, R' \in \mathcal{L}$, define $d(R, R') = |R \setminus R'|$. d is called the *Kemeny distance* (Kemeny, 1959; Kemeny and Snell, 1962).

I define properties of social welfare functions.

Independence of irrelevant alternatives (Arrow, 1951, 1963) requires that for each pair of alternatives x and y, the social preference over x and y depends only on the agents' preferences restricted to $\{x, y\}$.

Independence of Irrelevant Alternatives, IIA: For each $\boldsymbol{R}, \boldsymbol{R}' \in \mathcal{D}^N$, each pair $x, y \in X$,

$$\boldsymbol{R}|\{x, y\} = \boldsymbol{R}'|\{x, y\} \Rightarrow f(\boldsymbol{R})|\{x, y\} = f(\boldsymbol{R}')|\{x, y\}.$$

The next property appears only in the proof of a main result. This property is a weaker version of *IIA*. At $\boldsymbol{R}$, assume that agent i changes his preference from R_i to $R_i^{x,y}$ for some adjacent alternatives x and y. By *IIA*, for each $z, w \in X$ such that $\{z, w\} \neq \{x, y\}$, the social preference over z and w does not change. The following property says that this is the only requirement.

A-IIA[12]**:** For each $\boldsymbol{R} \in \mathcal{D}^N$, each $i \in N$, each $x, y \in X$ such that x and y are adjacent in R_i, if $R_i^{x,y} \in \mathcal{D}$,

$$f(\boldsymbol{R}) \setminus \{(x, y), (y, x)\} = f(R_i^{x,y}, \boldsymbol{R}_{-i}) \setminus \{(x, y), (y, x)\}. \tag{2.1}$$

The next property, *smooth response*, says that the smallest change of the variable (i.e., preference profile) leads to the smallest change, if any, of the value (i.e., social preference). The degree of changes is measured by the Kemeny distance.

Smooth response: For each $\boldsymbol{R} \in \mathcal{D}^N$, each $i \in N$, and each $R_i' \in \mathcal{D}$,

$$d(R_i, R_i') = 1 \Rightarrow d(f(\boldsymbol{R}), f(R_i', \boldsymbol{R}_{-i})) \leq 1.$$

[12] "A" stands for "Adjacent".

The next property, called *BM-monotonicity*[13], is used only for the discussion on Blair and Muller (1983). *BM-monotonicity* says that if x is socially preferred to y at some profile, then agents' additional supports for x never reverse the social preference over x and y.

BM-monotonicity: For each $\boldsymbol{R}, \boldsymbol{R}' \in \mathcal{D}^N$, and each $x, y \in X$ such that $\{i \in N \mid xR_iy\} \subsetneq \{i \in N \mid xR'_iy\}$,

$$xf(\boldsymbol{R})y \Rightarrow xf(\boldsymbol{R}')y.$$

In $\boldsymbol{R}$ and $\boldsymbol{R}'$ in the definition of *BM-monotonicity*, there is no constraint on preferences over alternatives in $X \setminus \{x, y\}$. Thus, it has a flavor of *IIA*. By restricting the scope of application of *BM-monotonicity*, we have the following weaker notion of *monotonicity*. These two notions are not equivalent. However, under *IIA* and a domain condition which will be described shortly, our main theorem implies that *BM-monotonicity* and *monotonicity* are equivalent.

Monotonicity: For each $\boldsymbol{R} \in \mathcal{D}^N$, each $i \in N$, and each adjacent alternatives $x, y \in X$ in R_i such that yR_ix,

$$xf(\boldsymbol{R})y \Rightarrow xf(R_i^{x,y}, \boldsymbol{R}_{-i})y.$$

We assume that agent i with the sincere preference R_i misreport a false preference R'_i if the misrepresentation leads to the social preference *more accordant with* R_i. We say R' *is more accordant with* R_i *than* R if the intersection of R_i and R is strictly contained to the intersection of R_i and R', i.e., $(R \cap R_i) \subsetneq (R' \cap R_i)$. When (x, y) belongs to the intersection of R_i and R, it means that in both R_i and R, x is preferred to y. Thus, R_i and R agree on the preference over x and y. Then, R' is more accordant with R_i than R if the agreement with R_i expands from R to R'. Some rational agent might find it profitable to misreport his preferences even

[13] "BM" stands for Blair and Muller (1983).

if the social welfare function is *nonmanipulable* in our sense. Nevertheless, our *nonmanipulability* plays a role to distinguish a desirable rule from an undesirable rule from the viewpoint of giving agents an incentive to report sincere preferences.

Manipulability: There are $\boldsymbol{R} \in \mathcal{D}^N$, $i \in N$, and $R_i' \in \mathcal{D}$ such that

$$(f(\boldsymbol{R}) \cap R_i) \subsetneq \left(f(R_i', \boldsymbol{R}_{-i}) \cap R_i\right).$$

Nonmanipulability: Not *manipulable*.

By restricting the options of misrepresentation to the preferences adjacent to the sincere one, we have the following property. There is a rationale to consider such a restriction. See Sato (2012) for details.

AP-manipulability[14]**:** There are $\boldsymbol{R} \in \mathcal{D}^N$, $i \in N$, and $x, y \in X$ such that

$$(f(\boldsymbol{R}) \cap R_i) \subsetneq (f(R_i^{x,y}, \boldsymbol{R}_{-i}) \cap R_i).$$

AP-nonmanipulability: Not *AP-manipulable*.

I also consider *strategy-proofness* of each derived social choice function. Blair and Muller (1983) show that each social welfare function satisfies *BM-monotonicity* and *IIA* if and only if each derived social choice function is *strategy-proof*. The equivalent formulation of *strategy-proofness* of each derived social choice function is the following.

SCF-manipulability[15]**:** There are $\boldsymbol{R} \in \mathcal{D}^N$, $i \in N$, $R_i' \in \mathcal{D}$, and $A \subset X$ such that

$$r_1(f(R_i', \boldsymbol{R}_{-i}), A) P_i r_1(f(\boldsymbol{R}), A).$$

SCF-nonmanipulability: Not *SCF-manipulable*.

The reader can see how strong *SCF-nonmanipulability* is by the following example. Let R_i in Table 1 be the agent i's true preference. If the social preference

[14] "AP" stands for "Adjacent Preferences".

[15] "SCF" stands for "Social Choice Functions".

Table 1: SCF-manipulability

R_i	R	$\bar{R}$
x	x	z
y	y	w
z	w	y
w	z	x

Table 2: Adjacent preferences and a path

R_i^1	R_i^2	R_i^3
x	x	z
y	z	x
z	y	y

changes from R to $\bar{R}$ by misrepresentation, then it is *SCF-manipulable*. Also, if the social preference changes from $\bar{R}$ to R by misrepresentation, then it is also *SCF-manipulable*.

To describe a condition on domains in our main result, I need several definitions. For each $R_i, R_i' \in \mathcal{D}$, R_i and R_i' are *adjacent* if R_i can be obtained by exchanging the positions of one pair of adjacent alternatives, $r_k(R_i')$ and $r_{k+1}(R_i')$ for some $k \in \{1, \ldots, m-1\}$, in R_i'. In this case, we write $A(R_i, R_i')$ for $\{r_k(R_i'), r_{k+1}(R_i')\}$. For example, in Table 2, R_i^1 and R_i^2 are adjacent and $A(R_i^1, R_i^2) = \{y, z\}$.

For each $R_i, R_i' \in \mathcal{D}$, a sequence $(R_i^1 R_i^2 \ldots R_i^\ell)$ is *a path in $\mathcal{D}$ from R_i to R_i'* if

(i) for each $k \in \{1, \ldots, \ell\}$, $R_i^k \in \mathcal{D}$,

(ii) $R_i^1 = R_i$ and $R_i^\ell = R_i'$,

(iii) for each $k \in \{1, \ldots, \ell - 1\}$, R_i^k is adjacent to R_i^{k+1}, and

(iv) for each distinct $k, h \in \{1, \ldots, \ell - 1\}$, $A(R_i^k, R_i^{k+1}) \neq A(R_i^h, R_i^{h+1})$.

In Table 2, $(R_i^1 R_i^2 R_i^3)$ is a path from R_i^1 to R_i^3. However, $(R_i^1 R_i^2 R_i^3 R_i^2)$ is not a path from R_i^1 to R_i^2.

Definition 2.1

A domain $\mathcal{D}$ is *connected* if for each $R_i, R_i' \in \mathcal{D}$, there is a path in $\mathcal{D}$ from R_i to R_i'.

The universal domain $\mathcal{L}$ and the set of all single-peaked preferences are examples of *connected* domains (Sato, 2012).

3 Result

3.1 A theorem

As a main result, I establish the equivalence of four statements.

Theorem 3.1

Let f be a social welfare function on a connected domain $\mathcal{D}$. The following statements are equivalent:

(i) f satisfies smooth response *and* nonmanipulability.

(ii) f satisfies smooth response *and* AP-nonmanipulability.

(iii) f satisfies monotonicity *and* IIA.

(iv) f satisfies SCF-nonmanipulability.

From this equivalence theorem, we have the implications discussed in the introduction: implications of *IIA*, *nonmanipulability* of social welfare functions, implications of *strategy-proofness* of each derived social choice functions, the significance of *connectedness* of domains, and implications of *smooth response*.

Several remarks are in order.

First, we could include more statements in the theorem. For example, "f satisfies *monotonicity* and *A-IIA*" is equivalent to each statement in the theorem. I exclude this statement because I am not sure about the importance of *A-IIA*.

Second, some direction of the implications is trivial. For example, (i) $\Rightarrow$ (ii) follows by definitions.

Third, the equivalence (iii) and (iv) is not the same as Blair and Muller (1983). (Our definition of *monotonicity* is weaker than *BM-monotonicity*.)

3.2 Proofs

In the following, I prove the theorem. Let f be a social welfare function on a connected domain $\mathcal{D}$.

Lemma 3.1

If f satisfies smooth response *and* AP-nonmanipulability, *for each $\boldsymbol{R} \in \mathcal{D}^N$, each $i \in N$, and each $x, y \in X$ such that $R_i^{x,y} \in \mathcal{D}$ and yR_ix, either*

- $f(R_i^{x,y}, \boldsymbol{R}_{-i}) = f(\boldsymbol{R})$, *or*
- $yf(\boldsymbol{R})x$ *and* $xf(R_i^{x,y}, \boldsymbol{R}_{-i})y$, *and for each* $u, v \in X$ *such that* $\{u, v\} \neq \{x, y\}$, $f(R_i^{x,y}, \boldsymbol{R}_{-i})|\{u, v\} = f(\boldsymbol{R})|\{u, v\}$.

Proof. By *smooth response*, either $f(R_i^{x,y}, \boldsymbol{R}_{-i}) = f(\boldsymbol{R})$, or we have $f(R_i^{x,y}, \boldsymbol{R}_{-i})$ by exchanging the positions of one pair of adjacent alternatives in $f(\boldsymbol{R})$. Assume $f(R_i^{x,y}, \boldsymbol{R}_{-i}) \neq f(\boldsymbol{R})$.

I claim that the adjacent alternatives whose ranks are exchanged in the passage from $f(\boldsymbol{R})$ to $f(R_i^{x,y}, \boldsymbol{R}_{-i})$ are x and y, and $yf(\boldsymbol{R})x$ and $xf(R_i^{x,y}, \boldsymbol{R}_{-i})y$. (The situation is described in 3.) To show this, I derive a contradiction in all the other cases.

- CASE 1: The alternatives whose ranks are exchanged in the passage from $f(\boldsymbol{R})$ to $f(R_i^{x,y}, \boldsymbol{R}_{-i})$ are z and w such that $\{z, w\} \neq \{x, y\}$.

Table 3: The possible change of social preferences

R_i	$R_i^{x,y}$	$f(\boldsymbol{R})$	$f(R_i^{x,y}, \boldsymbol{R}_{-i})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
y	x	y	x
x	y	x	y
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4: Social preferences in Cases 1 and 2

Case 1		Case 2	
$f(\boldsymbol{R})$	$f(R_i^{x,y}, \boldsymbol{R}_{-i})$	$f(\boldsymbol{R})$	$f(R_i^{x,y}, \boldsymbol{R}_{-i})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
z	w	x	y
w	z	y	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- CASE 2: The alternative whose ranks are exchanged in the passage from $f(\boldsymbol{R})$ to $f(R_i^{x,y}, \boldsymbol{R}_{-i})$ are x and y, and $xf(\boldsymbol{R})y$ and $yf(R_i^{x,y}, \boldsymbol{R}_{-i})x$.

Consider Case 1 (Table 4). Without loss of generality, assume $zf(\boldsymbol{R})w$. There are two subcases: zR_iw or wR_iz. First, assume zR_iw. Because $\{z, w\} \neq \{x, y\}$, $zR_i^{x,y}w$. Then, $f(\boldsymbol{R})$ is more accordant with $R_i^{x,y}$ than $f(R_i^{x,y}, \boldsymbol{R}_{-i})$, which is a contradiction to *AP-nonmanipulability*. Next, assume wR_iz. Then, $f(R_i^{x,y}, \boldsymbol{R}_{-i})$ is more accordant with R_i than $f(\boldsymbol{R})$, which is a contradiction.

Next, consider case 2. Because yR_ix, $f(R_i^{x,y}, \boldsymbol{R}_{-i})$ is more accordant with R_i than $f(\boldsymbol{R})$. This is a contradiction to *AP-nonmanipulability*.

Thus, we can conclude that the difference between $f(\boldsymbol{R})$ and $f(R_i^{x,y}, \boldsymbol{R}_{-i})$ is the preference over x and y. Moreover, $yf(\boldsymbol{R})x$ and $xf(R_i^{x,y}, \boldsymbol{R}_{-i})y$. ■

Table 5: Nonexistence of x^* and y^*

R_i	$f(\boldsymbol{R})$	$f(R'_i, \boldsymbol{R}_{-i})$
$\vdots$	$\vdots$	$\vdots$
y^*	x^*	y^*
$\vdots$	$\vdots$	$\vdots$
x^*	y^*	x^*
$\vdots$	$\vdots$	$\vdots$

Lemma 3.2

Under smooth response*, the following statements are equivalent:*

(i) f *satisfies* nonmanipulability*.*

(ii) f *satisfies* AP-nonmanipulability*.*

Proof. (i) $\Rightarrow$ (ii) follows from the definitions.

(ii) $\Rightarrow$ (i). Let $\boldsymbol{R} \in \mathcal{D}^N$, $i \in N$, and $R'_i \in \mathcal{D}$. Let $(R_i^1 \dots R_i^\ell)$ be a path in $\mathcal{D}$ from R_i to R'_i. To show that f is *nonmanipulable*, it suffices to show that there are no $x^*, y^* \in X$ such that $y^* R_i x^*$, $x^* f(\boldsymbol{R}) y^*$, and $y^* f(R'_i, \boldsymbol{R}_{-i}) x^*$. (The situation is described in Table 5.) I prove this by induction.

INDUCTION BASE: There are no $x^*, y^* \in X$ such that $y^* R_i x^*$, $x^* f(\boldsymbol{R}) y^*$, and $y^* f(R_i^1, \boldsymbol{R}_{-i}) x^*$. This follows from $R_i = R_i^1$.

Let $k \in \{1, \dots, \ell - 1\}$.

INDUCTION HYPOTHESIS: There are no $x^*, y^* \in X$ such that $y^* R_i x^*$, $x^* f(\boldsymbol{R}) y^*$, and $y^* f(R_i^k, \boldsymbol{R}_{-i}) x^*$.

INDUCTION STEP: There are no $x^*, y^* \in X$ such that $y^* R_i x^*$, $x^* f(\boldsymbol{R}) y^*$, and $y^* f(R_i^{k+1}, \boldsymbol{R}_{-i}) x^*$.

In the following, I prove the induction step. Let $A(R_i^k, R_i^{k+1}) = \{x, y\}$. Without loss of generality, assume $y R_i^k x$ and $x R_i^{k+1} y$. By Lemma 3.1, in the passage

from $f(R_i^k, \boldsymbol{R}_{-i})$ to $f(R_i^{k+1}, \boldsymbol{R}_{-i})$, the only possible change in the social preference is $yf(R_i^k, \boldsymbol{R}_{-i})x$ and $xf(R_i^{k+1}, \boldsymbol{R}_{-i})y$. By the induction hypothesis, the only candidate for $\{x^*, y^*\}$ is $\{x, y\}$. Because $(R_i^1 \dots R_i^\ell)$ is a path, there is no $h \in \{1, \dots, k-1\}$ such that $A(R_i^h, R_i^{h+1}) = \{x, y\}$. Thus, at $R_i^1 = R_i$, the preference over x and y should be yR_ix. Therefore, $R_i|\{x, y\} \neq f(R_i^{k+1}, \boldsymbol{R}_{-i})|\{x, y\}$. This implies that $\{x, y\}$ cannot play the role of $\{x^*, y^*\}$. Therefore, there are no $x^*, y^* \in X$ such that $y^*R_ix^*$, $x^*f(\boldsymbol{R})y^*$, and $y^*f(R_i^{k+1}, \boldsymbol{R}_{-i})x^*$.

By the induction on k, there are no $x^*, y^* \in X$ such that $y^*R_ix^*$, $x^*f(\boldsymbol{R})y^*$, and $y^*f(R_i', \boldsymbol{R}_{-i})x^*$. This implies that f is *nonmanipulable*. ■

Lemma 3.3

The following statements are equivalent:

(i) f *satisfies* smooth response *and* AP-nonmanipulability

(ii) f *satisfies* monotonicity *and* A-IIA.

Proof. (i) ⇒ (ii) follows from Lemma 3.1.

(ii) ⇒ (i). Assume *monotonicity* and *A-IIA*. *Smooth response* follows from *A-IIA*. To prove *AP-nonmanipulability*, let $\boldsymbol{R} \in \mathcal{D}^N$, $i \in N$, and $x, y \in X$ such that $R_i^{x,y} \in \mathcal{D}$. By *A-IIA*, $f(\boldsymbol{R}) \setminus \{(x, y), (y, x)\} = f(R_i^{x,y}, \boldsymbol{R}_{-i}) \setminus \{(x, y), (y, x)\}$. Without loss of generality, assume yR_ix. If $xf(\boldsymbol{R})y$, by *monotonicity*, $xf(R_i^{x,y}, \boldsymbol{R}_{-i})y$, and $f(\boldsymbol{R}) = f(R_i^{x,y}, \boldsymbol{R}_{-i})$. If $yf(\boldsymbol{R})x$, regardless of $f(R_i^{x,y}, \boldsymbol{R}_{-i})|\{x, y\}$, $(f(\boldsymbol{R}) \cap R_i) \subsetneq (f(R_i^{x,y}, \boldsymbol{R}_{-i}) \cap R_i)$ does not hold. In each case, $f(R_i^{x,y}, \boldsymbol{R}_{-i})$ is not more accordant with R_i than $f(\boldsymbol{R})$. ■

Lemma 3.4

The following statements are equivalent:

(i) f *satisfies* A-IIA.

(ii) f *satisfies* IIA.

Proof. (ii) $\Rightarrow$ (i) follows from the definitions.

(i) $\Rightarrow$ (ii). Assume that f satisfies *A-IIA*. Let $x, y \in X$ and $\boldsymbol{R}, \boldsymbol{R}' \in \mathcal{D}$ be such that $\boldsymbol{R}|\{x,y\} = \boldsymbol{R}'|\{x,y\}$. The goal of the proof is $f(\boldsymbol{R})|\{x,y\} = f(\boldsymbol{R}')|\{x,y\}$.

Let $i \in N$. I prove $f(\boldsymbol{R})|\{x,y\} = f(R_i', \boldsymbol{R}_{-i})|\{x,y\}$. Because $\mathcal{D}$ is *connected*, there is a path $(R_i^1 \dots R_i^\ell)$ in $\mathcal{D}$ from R_i to R_i'. Because $R_i|\{x,y\} = R_i'|\{x,y\}$, there is no $k \in \{1, \dots, \ell - 1\}$ such that $A(R_i^k, R_i^{k+1}) = \{x,y\}$. (If there are such R_i^k and R_i^{k+1}, there should be $h \in \{1, \dots, \ell - 1\}$ with $h \neq k$ such that $A(R_i^h, R_i^{h+1}) = \{x,y\}$. This is a contradiction to the definition of a path.)

Let $k \in \{1, \dots, \ell - 1\}$. Because $A(R_i^k, R_i^{k+1}) \neq \{x,y\}$, by *A-IIA*, $f(R_i^k, \boldsymbol{R}_{-i})|\{x,y\} = f(R_i^{k+1}, \boldsymbol{R}_{-i})|\{x,y\}$. (Remember that *A-IIA* implies that the only possible change in the passage from $f(R_i^k, \boldsymbol{R}_{-i})$ to $f(R_i^{k+1}, \boldsymbol{R}_{-i})$ is the positions of alternatives in $A(R_i^k, R_i^{k+1})$. Thus, $f(R_i, \boldsymbol{R}_{-i})|\{x,y\} = f(R_i^1, \boldsymbol{R}_{-i})|\{x,y\} = f(R_i^2, \boldsymbol{R}_{-i})|\{x,y\} = \dots = f(R_i^\ell, \boldsymbol{R}_{-i})|\{x,y\} = f(R_i', \boldsymbol{R}_{-i})|\{x,y\}$.

We have shown that when agent i changes his preferences from R_i to R_i' at $\boldsymbol{R}$, the social preference over x and y does not change. By the same arguments, we can show that at $(R_i', \boldsymbol{R}_{-i})$, when agent $j \in N \setminus \{i\}$ changes his preference from R_j to R_j', the social preference over x and y does not change.

By repeating the arguments, we can conclude $f(\boldsymbol{R})|\{x,y\} = f(\boldsymbol{R}')|\{x,y\}$. ■

Lemma 3.5

On a connected domain $\mathcal{D}$, the following statements are equivalent:

(i) f satisfies smooth response *and* AP-nonmanipulability.

(ii) f satisfies SCF-nonmanipulability.

Proof. (i) $\Rightarrow$ (ii). Assume that f satisfies *smooth response* and *AP-nonmanipulability*. Suppose that f violates *SCF-nonmanipulability*. Then, there are $\boldsymbol{R} \in \mathcal{D}^N$, $i \in N$, $R_i' \in \mathcal{D}$, and $A \subset X$ such that $r_1(f(R_i', \boldsymbol{R}_{-i}), A) P_i r_1(f(\boldsymbol{R}), A)$. Let $x = r_1(f(\boldsymbol{R}), A)$ and $y = r_1(f(R_i', \boldsymbol{R}_{-i}), A)$. By the definition, $y R_i x$, $x f(\boldsymbol{R}) y$, and $y f(R_i', \boldsymbol{R}_{-i}) x$.

Let $(R_i^1 \dots R_i^\ell)$ be a path from R_i to R_i'. There are two cases.

CASE 1: $yR_i'x$. Because $(R_i^1 \dots R_i^\ell)$ is a path and yR_i^1x and $yR_i^\ell x$, by the definition, for each $k \in \{1, \dots, \ell\}$, $yR_i^k x$. By Lemma 3.1, along the path, the social preferences over x and y never change, i.e., $xf(R_i^k, \boldsymbol{R}_{-i})y$ for each $k \in \{1, \dots, \ell\}$. Thus, $f(R_i', \boldsymbol{R}_{-i})$, which is a contradiction to the assumption.

CASE 2: $xR_i'y$. Because $(R_i^1 \dots R_i^\ell)$ is a path, there is only one $k \in \{1, \dots, \ell - 1\}$ such that $yR_i^k x$ and $xR_i^{k+1}y$. By Lemma 3.1, along the path, the social preferences over x and y never change. Because $xf(\boldsymbol{R})y$, by Lemma 3.1, $xf(R_i^\ell, \boldsymbol{R}_{-i})y$. This is a contradiction.

(ii) $\Rightarrow$ (i). Assume that f satisfies *SCF-nonmanipulability*.

Suppose that f violates *smooth response*. Then, there are $\boldsymbol{R} \in \mathcal{D}^N$, $i \in N$, $x, y \in X$ such that $R_i^{x,y} \in \mathcal{D}$ and $wf(\boldsymbol{R})z$ and $zf(R_i^{x,y}, \boldsymbol{R}_{-i})w$ for some $z, w \in X$ with $\{z, w\} \neq \{x, y\}$. (If there are no such z and w, the only difference between $f(\boldsymbol{R})$ and $f(R_i^{x,y}, \boldsymbol{R}_{-i})$ is the preference over x and y. Thus, the Kemeny distance between them is one, which does not imply the violation of *smooth response*.) Let $A = \{z, w\}$. If zR_iw, then $r_1(f(R_i^{x,y}, \boldsymbol{R}_{-i}), A)P_ir_1(f(\boldsymbol{R}), A)$, which is a contradiction to *SCF-nonmanipulability*. If wR_iz, then $wR_i^{x,y}z$. Thus, $r_1(f(\boldsymbol{R}), A)P_i^{x,y}r_1(f(R_i^{x,y}, \boldsymbol{R}_{-i}), A)$. This is a contradiction to *SCF-nonmanipulability*. Therefore, f satisfies *smooth response*.

Suppose that f violates *AP-nonmanipulability*. Then, there are $\boldsymbol{R} \in \mathcal{D}^N$, $i \in N$, $x, y \in X$ such that $R_i^{x,y} \in \mathcal{D}$ such that $(f(\boldsymbol{R}) \cap R_i) \subsetneq (f(R_i^{x,y}, \boldsymbol{R}_{-i}) \cap R_i)$. Let $(z, w) \in (f(R_i^{x,y}, \boldsymbol{R}_{-i}) \cap R_i) \setminus (f(\boldsymbol{R}) \cap R_i)$. Let $A = \{z, w\}$. Then, $z = r_1(f(R_i^{x,y}, \boldsymbol{R}_{-i}), A)P_ir_1(f(\boldsymbol{R}), A) = w$. This is a contradiction to *SCF-nonmanipulability*. ■

Lemmas 3.2, 3.3, 3.4, and 3.5 establish the theorem.

4 Concluding remarks

My main contribution is a new equivalence of *IIA* and *nonmanipulability*. Moreover, we show the applicability of Sato (2012)'s technique to show the equivalence of several properties. *Smooth response* also plays an important rule. As I discussed in the introduction, *smooth response* is a natural property, but it is not a normative one. The case without *smooth response* would be a subject of future research.

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