CAES Working Paper Series

An impossibility for strategy-proof and majority matching rules

Tsuyoshi Adachi

Faculty of Economics, Takasaki City University of Economics

Takumi Kongo

Faculty of Economics, Fukuoka University

WP-2016-004

Center for Advanced Economic Study

Fukuoka University

(CAES)

8-19-1 Nanakuma, Jonan-ku, Fukuoka,
JAPAN 814-0180

An impossibility for strategy-proof and majority matching rules

Tsuyoshi Adachi[*] and Takumi Kongo[†]

May 18, 2016

Abstract

We show that strategy-proofness and majority decision making are incompatible in two-sided matching problems. The majority decision corresponds to a concept weaker than stability and stronger than (Pareto) efficiency. Since it is well-known that strategy-proofness is incompatible with stability and compatible with efficiency, our result can not only be seen as a generalization of the celebrated incompatibility between strategy-proofness and stability, but also as a clarification of the difficulty of strengthening efficiency alongside retaining strategy-proofness. A crucial difference between our incompatibility and two recent generalizations of the incompatibility between strategy-proofness and stability (Takagi & Serizawa 2010, Social Choice and Welfare 35:245-266, and Kongo 2013, Social Choice and Welfare 40:461-478) is that the majority decision does not focus on the mutually best-preferred pair. Such a pair always forms a blocking pair if they are not matched with each other, thus respecting the pairs is inherent in stability. However, the majority decision treats such blocking pairs and agents left behind by the blocking pairs equally. From this aspect, the majority decision can be perceived as being closer to efficiency than stability. Nonetheless, the majority decision and strategy-proofness eventually end in incompatibility. We also prove that strategy-proofness and the concept of minimizing choice numbers, which is logically independent of stability, are incompatible.

Keywords: matching, strategy-proofness, stability, majority, minimizing choice numbers, impossibility theorem

JEL classification: C78, D61

1 Introduction

One of the central solution concepts in two-sided matching problems (Gale and Shapley 1962) is stability. Stability requires two conditions. First, no agent can be matched

[*]Faculty of Economics, Takasaki City University of Economics, 1300 Kaminamie, Takasaki, Gunma 370-0801, Japan

[†]Corresponding author. Faculty of Economics, Fukuoka University, 8-19-1 Nanakuma, Jonan-ku, Fukuoka 814-0180 Japan, E-mail: kongo@adm.fukuoka-u.ac.jp, Tel: +81-92-871-6631.

with any unwanted agent (i.e., individual rationality). Second, no pair of agents can be better off by being matched with each other (i.e., no blocking pairs). Stability can thus be interpreted as the property that no one is coerced into being matched with another agent, and thus, it is important for agents to accept the outcomes. This can be confirmed by the fact that rules producing stable outcomes tend to survive; on the other hand, some of those producing unstable outcomes have been abandoned (see Roth 1991, 2002).

Despite its importance, however, stability is incompatible with another important property of strategy-proofness in two-sided matching problems (Theorem 3 in Roth 1982). Strategy-proofness requires that no agent has an incentive to misrepresent his or her true preference. According to this property, we can eliminate the possibility of disorder in the outcomes caused by the misrepresentation of preferences by a self-interested agent; hence, strategy-proofness is also important. However, as long as we pay attention to strategy-proofness, we are forced to abandon another important property of stability.

As an alternative to stability, (Pareto) efficiency is a weaker requirement that is compatible with strategy-proofness (Theorem 4 in Roth 1982). Efficiency requires that no agent is better off unless another agent is worse off. Although they are non-wasteful, efficient outcomes may enforce unacceptable outcomes for some agents, for instance, being matched with unwanted agents or not being matched with agents who share the most preferred mutual outcomes. Such agents may oppose accepting these outcomes.

The majority decision is one popular way to build a consensus among people with different opinions. In two-sided matching problems, Gärdenfors (1975) introduces a solution concept of majority matchings as an alternative to stability and efficiency. A majority matching requires that there is no other matching preferred by more agents. Once we admit a majority decision as a method of group decision making, agents who are not quite satisfied with the outcome in a majority matching are supposed to accept the matching, although unwillingly. Although they are less acceptable compared with stable matchings, majority matchings are more acceptable than efficient matchings. In fact, under the usual settings in two-sided matching problems, it has been shown that stable matchings are majority matchings that are efficient (Theorems 3 and 5 in Gärdenfors 1975). Then, together with the above-mentioned impossibility and possibility, a natural question arises: does any strategy-proof matching rule produce a majority matching?

We give a strong negative answer to this question. We show that even in pure matching problems (Tadenuma and Toda 1998) with three men and three women, strategy-proofness and majority matchings are incompatible. The pure matching problem is a one-to-one two-sided matching problem in which all agents want to match with all agents in the other set (i.e., they do not want to be single). In such kinds of limited problems, only three men and three women rule out the possibility of constructing the strategy-proof and majority rules. We also show that strategy-proofness and majority are compatible in problems with only two men and two women. Since the class of usual matching problems contains the class of pure matching problems, our result is valid for the class of usual matching problems when the cardinalities of both sets are more than or equal to three. Together with the findings of previous studies, our result thus clarifies the boundary between the impossibilities and possibilities of constructing strategy-proof rules in two-sided matching problems. Also, it sheds light on the difficulty of strengthening efficiency in conjunction with strategy-proofness.

With regard to our main result, we emphasize two things. First, our incompatibility is not a consequence of the impossibility results obtained by the previous seminal studies of Gibbard (1973) and Satterthwaite (1975). Any matching problem in which all agents have strict preferences on the agents in the other set can be converted into a corresponding social choice problem in which all agents have corresponding weak preferences on the set of outcomes (i.e., the set of possible matchings). However, in such a converted problem, the domain of weak preferences is restricted by the structures of the matchings. For example, consider pure matching problems between three men m_1, m_2, and m_3, and three women w_1, w_2, and w_3. In two of the six alternatives (possible pure matchings), m_1 and w_1 match with each other. Assuming that each agent is interested only in the mate he/she matches with, the two matchings are indifferent for m_1 and w_1. Meanwhile, the two matchings are not indifferent for the other agents, because they match with different agents in the two matchings, and thus they have strict preferences between the two. Therefore, the corresponding social choice problems are those with weak and restricted preferences, so that we cannot simply apply existing results on the domain of all weak preferences.

Second, the method of the proof for our main result is elaborate, and is not in the same manner as that for the celebrated impossibility result of Roth (1982). This feature of our proof is different from the proofs of generalizations of Roth's (1982) impossibility theorem on strategy-proof matching rules (referred to in the last but two paragraphs in the Introduction). Roughly speaking, in the proof of Roth's (1982) impossibility theorem, we focus on a specific preference profile, and then, we compare the set of stable matchings of the original preference profile with those of its adjacent preference profiles, each of which is induced by a manipulation of an agent from the original one. Strategy-proofness narrows down the possible stable matchings for the original and its adjacent preference profiles, and we obtain an incompatibility (see also Example 1 in Section 2). However, it seems that the above method is unsuitable for the case of majority matching, because specifying the set of majority matchings for preference profiles is rather harder than specifying the set of stable matchings. Hence, in our proof, we start from any problem satisfying the specific conditions on preference profiles. Then, we repeat procedures by (i) considering the full potential of the set of majority matchings for the target preference profile, (ii) shifting the target to its adjacent preference profile, and (iii) narrowing down the possible majority matchings for the shifted preference profile in conjunction with strategy-proofness. Using these procedures, we investigate the possibility of the set of majority matchings for some key preference profiles, each of which is obtained by sequences of adjacent preference profiles, and eventually, we obtain an incompatibility.

We also investigate the relationship between strategy-proofness and minimizing choice numbers, which is another approach to optimize matchings and also used by Gärdenfors (1975). For each agent and each of his/her preferences, the choice number of each member of the opposite sex is the number of members ranked before the member under the preferences. In other words, for each agent, the choice number of his/her first choice in his/her preferences is zero, that of his/her second choice is one, and so on. Based on the choice numbers of all agents, the choice number of each matching is defined as the sum of the choice numbers of all agents under the matching. This approach has no logical relationship with regard to the celebrated stability, and thus, it is often used for the selection of stable matchings (minimum choice stable solution or egalitarian stable matchings as in McVitie and Wilson 1971 and Irving et al. 1987). Meanwhile, Gärdenfors (1975) discusses matchings that minimize choice numbers among all possible matchings. We prove that no matching rule minimizes choice numbers and is strategy-proof for pure matching problems with three men and three women.

We refer to the literature related to our work, in other words, the literature focusing on the impossibilities of constructing strategy-proof matching rules. To the best of our knowledge, three kinds of different generalizations of Roth's (1982) incompatibility between stability and strategy-proofness have been discussed. First, Alcalde and Barberà (1994) show that there is no rule that is Pareto efficient, individually rational, and strategy-proof.[1] Second, Takagi and Serizawa (2010) introduce a new property that is weaker than stability. Their property is called respect for 2-unanimity. This property respects each agent's first choice if it is mutual.[2] Their property is shown to be incompatible with strategy-proofness. Third, as a parallel to Takagi and Serizawa's (2010) respect for 2-unanimity, Kongo (2013) defines a new property that is also weaker than stability. The property is called respect for recursive unanimity and respects all agents' first choices at some steps if they are mutual in a recursive manner. This property is also proven to be incompatible with strategy-proofness.

In the above literature, both Takagi and Serizawa (2010) and Kongo (2013) focus on the property of respecting mutually best-preferred pairs, which is inherent to stability because such kinds of pairs always form blocking pairs if they are not matched with each other. In regard to this point, majority matchings make a crucial difference as they do not always respect the mutually best-preferred pairs. Some majority matchings do not respect the mutually best-preferred blocking pairs if both mates in each pair matched under the original matching are against respecting the mutually best-preferred pair. Now, the number of agents who agree to respect the pair and that against it are the same (two), and hence, the majority decision requires nothing. In this sense, we say that the majority looks closer to efficiency than stability, and also, that our incompatibility between strategy-proofness and majority is not along the same lines as those in Takagi and Serizawa (2010) and Kongo (2013).

The rest of the paper is constructed as follows. Section 2 presents our model. Section 3 discusses our results.

2 Model

We focus on the rather restricted problem (i.e., the number of agents is fixed and their preferences are restricted) of one-to-one two-sided matching (Gale and Shapley 1962). Note that because our main result in the next section is an impossibility, it also holds in the class of typical two-sided matching problems.

[1]This result is further investigated in the two-sided many-to-one matching problems by Sönmez (1996).
[2]This property is closely related to a property of mutually best in Toda (2006).

Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ be mutually disjoint sets. For each $i \in M$ (resp. W), let $\boldsymbol{P}_i$ be the set of all total orders on W (resp. M). Let $P_i \in \boldsymbol{P}_i$ be *a (strict) preference of i*, and $P_i : x, y, z$ represents xP_iyP_iz in P_i. Here, we restrict our attention to so-called pure matching problems (Tadenuma and Toda 1998), where being unmatched is infeasible (or the worst alternative) for all agents. We call $P = (P_{m_1}, P_{m_2}, \dots, P_{w_3}) \in \prod_{M \cup W} \boldsymbol{P}_i (\stackrel{\text{def}}{=} \boldsymbol{P})$ *a preference profile*.

A *matching* is a collection of pairs of an element in M and that in W, where all elements in both sets appear once. Let $\{xX, yY, zZ\}$, where $\{x,y,z\}, \{X,Y,Z\} \in \{M, W\}$ and $\{x,y,z\} \neq \{X,Y,Z\}$ denote a matching, where x matches with X (i.e., x and X are paired), y matches with Y, and z matches with Z. Let $\mathcal{A}$ be the set of all matchings between M and W.

Because we restrict our attention to the pure matching problem between three men and three women, we have six possible matchings. We let $\mu_1 = \{m_1w_1, m_2w_2, m_3w_3\}$, $\mu_2 = \{m_1w_1, m_2w_3, m_3w_2\}$, $\mu_3 = \{m_1w_2, m_2w_1, m_3w_3\}$, $\mu_4 = \{m_1w_2, m_2w_3, m_3w_1\}$, $\mu_5 = \{m_1w_3, m_2w_1, m_3w_2\}$, and $\mu_6 = \{m_1w_3, m_2w_2, m_3w_1\}$. Further, we assume that all agents are interested only in the mate they are matched with. Thus, given i's preference P_i, we can consider its corresponding weak preferences R_i on matchings. We let P_i denote asymmetric parts and I_i represent symmetric parts of R_i.[3] For example, let $P_{m_1} : w_1, w_2, w_3$. Then, we write $R_{m_1} : \mu_1, \mu_2; \mu_3, \mu_4; \mu_5, \mu_6$, meaning that $\mu_1 I_{m_1} \mu_2$, $\mu_2 P_{m_1} \mu_3$, $\mu_3 I_{m_1} \mu_4$, $\mu_4 P_{m_1} \mu_5$, and $\mu_5 I_{m_1} \mu_6$.

For each $P \in \boldsymbol{P}$, a pair of a man $m \in M$ and a woman $w \in W$ *blocks a matching* μ under P if each of them prefer the other to the mate under μ (i.e., $wP_m\mu(m)$ and $mP_w\mu(w)$), where $\mu(i)$ denotes the agent who matches with i under μ. A matching μ *beats* another matching μ' under P if $\#\{i \in M \cup W | \mu P_i \mu'\} > \#\{i \in M \cup W | \mu' P_i \mu\}$, where # represents the cardinality of a set. A matching μ *dominates* another matching μ' under P if $\#\{i \in M \cup W | \mu P_i \mu'\} > \#\{i \in M \cup W | \mu' P_i \mu\} = 0$. A matching μ is *stable* under P if it is not blocked by any pair of a man and a woman under P.[4] A matching μ is *majority* under P if it is not beaten by any other matching under P. A matching μ is *efficient* under P if it is not dominated by any other matching under P. Let $\mathcal{S}(P)$, $\mathcal{M}(P)$, and $\mathcal{E}(P)$ be the sets of all stable, majority, and efficient matchings under P, respectively. Gale and Shapley (1962) demonstrates that $\mathcal{S}(P) \neq \emptyset$ for any $P \in \boldsymbol{P}$. Gärdenfors (1975) shows that $\mathcal{S}(P) \subseteq \mathcal{M}(P) \subseteq \mathcal{E}(P)$ for any $P \in \boldsymbol{P}$.

Let $f : \boldsymbol{P} \to \mathcal{A}$ be *a (matching) rule*. A rule f is *stable* if for each $P \in \boldsymbol{P}$, $f(P) \in \mathcal{S}(P)$. A rule f is *majority* if for each $P \in \boldsymbol{P}$, $f(P) \in \mathcal{M}(P)$. A rule f is *efficient* if for each $P \in \boldsymbol{P}$, $f(P) \in \mathcal{E}(P)$. A rule f is *strategy-proof(SP)* if for each $P \in \boldsymbol{P}$, each $i \in M \cup W$, and each $P'_i \in \boldsymbol{P}_i$, $f(P) R_i f(P'_i, P_{-i})$. Roth (1982) proves that stability and SP are incompatible, and efficiency and SP are compatible.

The following example illustrates the distinction between stable, majority, and efficient matchings. It also illustrates that the proof of Theorem 3 in Roth (1982) is not enough for showing the incompatibility between the majority and SP rules.

[3] For simplicity, we use the same notation P_i for both preferences on the members of the opposite sex and those on matchings.

[4] Because we restrict our attention to pure matching problems, individual rationality is automatically satisfied.

Example 1. *Consider the following preference profile.*[5]

$$
\begin{aligned}
P_{m_1} &: w_1, w_2, w_3 \\
P_{m_2} &: w_2, w_1, w_3 \\
P_{m_3} &: w_2, w_3, w_1 \\
P_{w_1} &: m_2, m_1, m_3 \\
P_{w_2} &: m_1, m_2, m_3 \\
P_{w_3} &: m_2, m_3, m_1 \\
P'_{m_2} &: w_2, w_3, w_1 \\
P'_{w_2} &: m_1, m_3, m_2.^6
\end{aligned}
$$

Now, $\mathcal{S}(P) = \{\mu_1, \mu_3\}$, $\mathcal{M}(P) = \{\mu_1, \mu_2, \mu_3\}$, and $\mathcal{E}(P) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$. This is because no pair blocks μ_1 and μ_3, the pair of m_1 and w_1 blocks μ_4 and μ_6, the pair of m_2 and w_1 blocks μ_2, and the pair of m_2 and w_2 blocks μ_5. Table 1 shows the beat relationships between two distinct matchings under P. In the table, agents listed without parentheses in each cell prefer a row matching to a column matching, those written inside parentheses prefer a column matching to a row matching, and the unwritten agents are indifferent between the two matchings. For instance, the number of agents who prefer μ_4 to μ_6 is greater than that who prefer μ_6 to μ_4, and hence μ_4 beats μ_6. Also, μ_1 beats μ_4 and μ_5. Further, there is no domination relationship except μ_1 dominates μ_6.

This preference profile essentially corresponds to that in Theorem 3 in Roth (1982), which shows incompatibility between stability and SP. Now, $\mathcal{S}(P'_{w_2}, P_{-w_2}) = \{\mu_3\}$ and $\mathcal{S}(P'_{m_2}, P_{-m_2}) = \{\mu_1\}$. Since $\mu_3 P_{w_2} \mu_1$ and $\mu_1 P_{m_2} \mu_3$, any stable rule is not SP under P. Meanwhile, $\mathcal{M}(P'_{w_2}, P_{-w_2}) = \{\mu_2, \mu_3\}$, and $\mathcal{M}(P'_{m_2}, P_{-m_2}) = \{\mu_1, \mu_2\}$ (in both cases, μ_2 still remains). Because $\mu_3 P_{w_2} \mu_1 P_{w_2} \mu_2$ and $\mu_1 P_{m_2} \mu_3 P_{m_2} \mu_2$, these are not enough for showing an incompatibility between the majority and SP matchings.

Apart from the weakening stability approach, Gärdenfors (1975) also introduces an alternative solution concept of minimizing choice number matchings. Given $i \in M \cup W$,

[5] This preference profile corresponds to P^{cC} in Lemma 1 below.

[6] The following preferences on the set of matchings, corresponding to the above preferences on the agents on the other sides, are of great help to recognize which matchings are majority or efficient.

$$
\begin{aligned}
R_{m_1} &: \mu_1, \mu_2; \mu_3, \mu_4; \mu_5, \mu_6 \\
R_{m_2} &: \mu_1, \mu_6; \mu_3, \mu_5; \mu_2, \mu_4 \\
R_{m_3} &: \mu_2, \mu_5; \mu_1, \mu_3; \mu_4, \mu_6 \\
R_{w_1} &: \mu_3, \mu_5; \mu_1, \mu_2; \mu_4, \mu_6 \\
R_{w_2} &: \mu_3, \mu_4; \mu_1, \mu_6; \mu_2, \mu_5 \\
R_{w_3} &: \mu_2, \mu_4; \mu_1, \mu_3; \mu_5, \mu_6 \\
R'_{m_2} &: \mu_1, \mu_6; \mu_2, \mu_4; \mu_3, \mu_5 \\
R'_{w_2} &: \mu_3, \mu_4; \mu_2, \mu_5; \mu_1, \mu_6.
\end{aligned}
$$

Table 1: Agents preferences between two matchings under P

	μ_2	μ_3	μ_4	μ_5	μ_6
μ_1	m_2, w_2 (m_3, w_3)	m_1, m_2 (w_1, w_2)	m_1, m_2, m_3, w_1 (w_2, w_3)	m_1, m_2, w_2, w_3 (m_3, w_1)	m_1, m_3, w_1, w_3 (nobody)
μ_2		m_1, m_3, w_3 (m_2, w_1, w_2)	m_1, m_3, w_1 (w_2)	m_1, w_3 (m_2, w_1)	m_1, m_3, w_1, w_3 (m_2, w_2)
μ_3			m_2, m_3, w_1 (w_3)	m_1, w_2, w_3 (m_3)	m_1, m_3, w_1, w_2, w_3 (m_2)
μ_4				m_1, w_2, w_3 (m_2, m_3, w_1)	m_1, w_2, w_3 (m_2)
μ_5					m_3, w_1 (m_2, w_2)

$P_i \in \boldsymbol{P}_i$, and a matching μ, let $c(\mu(i),P_i) = \#\{j \in M \cup W \,|\, jP_i\mu(i)\}$ be a *choice number* for i under P_i. Let $\mathcal{C}(P) = \{\mu \in \mathcal{A} | \mu = \min_{\mu' \in \mathcal{A}} \sum_i c(\mu'(i),P_i)\}$ be a set of minimizing choice number matchings. A rule f minimizes choice numbers if for each $P \in \boldsymbol{P}$, $f(P) \in \mathcal{C}(P)$.

3 Results

Here, we show that there is no f that is both majority and SP. Since the tasks checking whether or not each matching is majority under each preference profile must be the hardest, we summarize the claims used in our proof, namely that these tasks are necessary to obtain, as one lemma; the lemma is composed of two claims used at technical points in our main proof.

Lemma 1. *Let $\{a,b,c\},\{A,B,C\} \in \{M,W\}$ and $\{a,b,c\} \neq \{A,B,C\}$. There are six possible matchings. We let $\nu_1 = \{aA,bB,cC\}$, $\nu_2 = \{aA,bC,cB\}$, $\nu_3 = \{aB,bA,cC\}$, $\nu_4 = \{aB,bC,cA\}$, $\nu_5 = \{aC,bA,cB\}$, and $\nu_6 = \{aC,bB,cA\}$. Consider the following preferences.*

$$
\begin{array}{ll}
P_a^0 : A,B,C & P_a^1 : A,C,B \\
P_b^0 : B,A,C & P_b^1 : B,C,A \\
P_c^0 : B,A,C & P_c^1 : B,C,A \\
P_A^0 : b,a,c & P_A^1 : b,c,a \\
P_B^0 : a,b,c & P_B^1 : a,c,b \\
P_C^0 : b,a,c & P_C^1 : b,c,a.^7
\end{array}
$$

[7] The following are the preferences on the set of matchings corresponding to the above preferences on the

For each $S \subseteq M \cup W$, let $P^S \in \boldsymbol{P}$ be such that

$$P_i^S = \begin{cases} P_i^1 & \text{if } i \in S \\ P_i^0 & \text{otherwise.}^8 \end{cases}$$

Let f refer to majority and SP. Then,
(i) If $f(P^{aBC}) \neq v_2$, then $f(P^{aC}) = f(P^{AC}) = v_3$.
(ii) Either $f(P^{bcC})$ or $f(P^{cAC})$ must be v_2.

The proof is composed of four steps. Steps 1 and 2 clarify the full potential of the set of majority matchings under some preference profiles. Based on these steps, Steps 3 and 4 prove Lemma 1 (i) and Lemma 1 (ii), respectively.

Proof of Lemma 1.

Step 1. (1) If $A \notin S$, then v_4 and v_6 are not in $\mathcal{M}(P^S)$.
(2) If $b \in S$, then v_3 and v_5 are not in $\mathcal{M}(P^S)$.
(3) If $B \in S$, then v_1 and v_6 are not in $\mathcal{M}(P^S)$.
(4) If $a \notin S$, then v_6 is not in $\mathcal{M}(P^S)$.
(5) If $A \in S$ and $c \notin S$, then v_1 is not in $\mathcal{M}(P^S)$.
(6) If $b \notin S$ and $c \in S$, then v_4 is not in $\mathcal{M}(P^S)$.
(7) If $C \in S$ and $\{a, B\} \nsubseteq S$, then v_5 is not in $\mathcal{M}(P^S)$.

Proof of Step 1. (1): v_2 beats both v_4 and v_6 under such P^S since for each $\hat{v} \in \{v_4, v_6\}$, $v_2 P_a^S \hat{v}, v_2 P_c^S \hat{v}, v_2 P_A^S \hat{v}$, and $v_2 R_C^S \hat{v}$.
(2): v_2 beats both v_3 and v_5 under such P^S since for each $\hat{v} \in \{v_3, v_5\}$, $v_2 P_a^S \hat{v}, v_2 P_b^S \hat{v}, v_2 R_c^S \hat{v}$, and $v_2 P_C^S \hat{v}$.
(3): v_2 beats both v_1 and v_6 under such P^S since for each $\hat{v} \in \{v_1, v_6\}$, $v_2 R_a^S \hat{v}, v_2 P_c^S \hat{v}, v_2 P_B^S \hat{v}$, and $v_2 P_C^S \hat{v}$.
(4): v_4 beats v_6 under such P^S since $v_4 P_a^S v_6, v_4 I_c^S v_6, v_4 P_B^S v_6$, and $v_4 P_C^S v_6$.
(5): v_4 beats v_1 under such P^S since $v_4 P_c^S v_1, v_4 P_A^S v_1, v_4 P_B^S v_1$, and $v_4 P_C^S v_1$.
(6): v_3 beats v_4 under such P^S since $v_3 P_b^S v_4, v_3 P_c^S v_4, v_3 P_A^S v_4$, and $v_3 I_B^S v_4$.
(7): If $a \notin S$, then v_3 beats v_5 under such P^S since $v_3 P_a^S v_5, v_3 I_b^S v_5, v_3 P_B^S v_5$, and $v_3 P_C^S v_5$.
If $B \notin S$, then v_1 beats v_5 under such P^S since $v_1 P_a^S v_5, v_1 P_b^S v_5, v_1 P_B^S v_5$, and $v_1 P_C^S v_5$. $\square$

---

agents on the other sides.

$$\begin{array}{ll}
R_a^0 : v_1, v_2; v_3, v_4; v_5, v_6 & R_a^1 : v_1, v_2; v_5, v_6; v_3, v_4 \\
R_b^0 : v_1, v_6; v_3, v_5; v_2, v_4 & R_b^1 : v_1, v_6; v_2, v_4; v_3, v_5 \\
R_c^0 : v_2, v_5; v_4, v_6; v_1, v_3 & R_c^1 : v_2, v_5; v_1, v_3; v_4, v_6 \\
R_A^0 : v_3, v_5; v_1, v_2; v_4, v_6 & R_A^1 : v_3, v_5; v_4, v_6; v_1, v_2 \\
R_B^0 : v_3, v_4; v_1, v_6; v_2, v_5 & R_B^1 : v_3, v_4; v_2, v_5; v_1, v_6 \\
R_C^0 : v_2, v_4; v_5, v_6; v_1, v_3 & R_C^1 : v_2, v_4; v_1, v_3; v_5, v_6.
\end{array}$$

[8]For simplicity, we write, for instance, P^{aBC} instead of writing $P^{\{a,B,C\}}$.

Step 2. It follows that

$$\begin{aligned}
\mathcal{M}(P^{C}),\mathcal{M}(P^{cC}),\mathcal{M}(P^{aC}),\mathcal{M}(P^{cAC}) \subseteq &\quad \{v_1,v_2,v_3\},\\
\mathcal{M}(P^{aBC}) \subseteq &\quad \{v_2,v_3,v_5\},\\
\mathcal{M}(P^{bcC}) \subseteq &\quad \{v_1,v_2\},\\
\mathcal{M}(P^{AC}),\mathcal{M}(P^{ABC}) \subseteq &\quad \{v_2,v_3,v_4\},\\
\mathcal{M}(P^{BC}),\mathcal{M}(P^{cABC}) \subseteq &\quad \{v_2,v_3\}.
\end{aligned}$$

Proof of Step 2. For $\mathcal{M}(P^{C}),\mathcal{M}(P^{cC})$, and $\mathcal{M}(P^{aC})$, Steps 1 (1) and (7) derive the conclusion. For $\mathcal{M}(P^{cAC})$, Steps 1 (4), (6), and (7) derive the conclusion. For $\mathcal{M}(P^{aBC})$, Steps 1 (1) and (3) derive the conclusion. For $\mathcal{M}(P^{bcC})$, Steps 1 (1) and (2) derive the conclusion. For $\mathcal{M}(P^{AC})$ and $\mathcal{M}(P^{ABC})$, Steps 1 (4), (5), and (7) derive the conclusion. For $\mathcal{M}(P^{BC})$, Steps 1 (1), (3), and (7) derive the conclusion. For $\mathcal{M}(P^{cABC})$, Steps 1 (3), (6), and (7) derive the conclusion. $\square$

Step 3. Let f be SP and majority. If $f(P^{aBC}) \neq v_2$, then $f(P^{aC}) = f(P^{AC}) = v_3$.

Proof of Step 3. Suppose that $f(P^{aBC}) \neq v_2$. By Step 2, $f(P^{aBC}) \in \{v_3,v_5\}$. By the SP for a, $f(P^{BC}) \neq v_2$. By Step 2, $f(P^{BC}) = v_3$. By the SP for B, $f(P^{C})$ is neither v_1 nor v_2. By Step 2, $f(P^{C}) = v_3$. By the SP for a and A, $f(P^{aC})$ is neither v_1 nor v_2, and $f(P^{AC})$ is neither v_2 nor v_4. By Step 2, $f(P^{aC}) = f(P^{AC}) = v_3$. $\square$

Step 4. Let f be SP and majority. Either $f(P^{bcC})$ or $f(P^{cAC})$ must be v_2.

Proof of Step 4. Suppose that both $f(P^{bcC})$ and $f(P^{cAC})$ are not v_2. By Step 2, $f(P^{bcC}) = v_1$. By the SP for b, $f(P^{cC})$ is neither v_3 nor v_2. By Step 2, $f(P^{cC}) = v_1$. By the SP for A, $f(P^{cAC})$ is not v_3. Since $f(P^{cAC}) \neq v_2$, by Step 2, $f(P^{cAC}) = v_1$. By the SP for B, $f(P^{cABC}) \neq v_3$. By Step 2, $f(P^{cABC}) = v_2$. By the SP for c, $f(P^{ABC})$ is neither v_3 nor v_4. By Step 2, $f(P^{ABC}) = v_2$. By the SP for B, $f(P^{AC})$ is neither v_3 nor v_4. By Step 2, $f(P^{AC}) = v_2$, which contradicts the fact that f is SP, since $f(P^{cAC}) = v_1$. $\square$

Steps 3 and 4 correspond to Lemmas 1 (i) and Lemma 1 (ii), respectively. $\square$

Now, we are ready to prove the main proposition. In the following proof, we start from any preference profile satisfying specific conditions. The full potential of the set of majority matchings under such a preference profile is investigated by the claim below. Then, by changing preferences one by one, we convert the profile into the ones considered in Lemma 1 (i), and further, we convert those into the ones considered in Lemma 1 (ii) to obtain a contradiction.

Proposition 1. *There is no f that is both SP and majority.*

Proof of Proposition 1. Let f be an SP and a majority rule. Let $\hat{\boldsymbol{P}}$ be the set of preference profiles such that $P \in \hat{\boldsymbol{P}}$ if and only if

$$\begin{aligned}
w_1 P_{m_1} w_2 \ \& \ w_1 P_{m_1} w_3,\\
w_2 P_{m_2} w_1 \ \& \ w_2 P_{m_2} w_3,\\
w_3 P_{m_3} w_1 \ \& \ w_3 P_{m_3} w_2,\\
m_2 P_{w_1} m_1 \ \& \ m_2 P_{w_1} m_3,\\
m_1 P_{w_2} m_2 \ \& \ m_1 P_{w_2} m_3,\\
m_3 P_{w_3} m_1 \ \& \ m_3 P_{w_3} m_2.^9
\end{aligned}$$

We establish the following claim.

Claim: For each $P \in \hat{\boldsymbol{P}}$, each $i \in \{m_1, m_2, w_1, w_2\}$, and each $P_i' \in \boldsymbol{P}_i$, it holds that $\mathcal{M}(P_i', P_{-i}) \subseteq \{\mu_1, \mu_3\}$.

Proof of the claim. If $i \in \{m_1, m_2\}$, then μ_3 beats μ_2, μ_4, μ_5, and μ_6 under (P_i', P_{-i}) since [$\mu_3 I_{w_1} \mu_5$, $\mu_3 P_{w_2} \mu_5$, $\mu_3 P_{m_3} \mu_5$, and $\mu_3 P_{w_3} \mu_5$], [$\mu_3 P_{w_1} \mu_4$, $\mu_3 I_{w_2} \mu_4$, $\mu_3 P_{m_3} \mu_4$, and $\mu_3 P_{w_3} \mu_4$], and [for each $\hat{\mu} \in \{\mu_2, \mu_6\}$, $\mu_3 P_{w_1} \hat{\mu}$, $\mu_3 P_{w_2} \hat{\mu}$, $\mu_3 P_{m_3} \hat{\mu}$, and $\mu_3 P_{w_3} \hat{\mu}$]. If $i \in \{w_1, w_2\}$, then μ_1 beats μ_2, μ_4, μ_5, and μ_6 under (P_i', P_{-i}) since [$\mu_1 I_{m_1} \mu_2$, $\mu_1 P_{m_2} \mu_2$, $\mu_1 P_{m_3} \mu_2$, and $\mu_1 P_{w_3} \mu_2$], [$\mu_1 P_{m_1} \mu_6$, $\mu_1 I_{m_2} \mu_6$, $\mu_1 P_{m_3} \mu_6$, and $\mu_1 P_{w_3} \mu_6$], and [for each $\hat{\mu} \in \{\mu_4, \mu_5\}$, $\mu_1 P_{m_1} \hat{\mu}$, $\mu_1 P_{m_2} \hat{\mu}$, $\mu_1 P_{m_3} \hat{\mu}$, and $\mu_1 P_{w_3} \hat{\mu}$]. □

Let $P \in \hat{\boldsymbol{P}}$. By the claim, $f(P)$ is either μ_1 or μ_3. We first consider the case where $f(P) = \mu_1$.

Let $P_{m_1}' : w_3, w_1, w_2$.[10] By the claim, $\mathcal{M}(P_{m_1}', P_{-m_1}) \subseteq \{\mu_1, \mu_3\}$. By the SP for m_1, $f(P_{m_1}', P_{-m_1}) = \mu_1$. Let $P_{m_2}' : w_2, w_3, w_1$.[11] Note that $(P_{m_2}', P_{-m_2}) \in \hat{\boldsymbol{P}}$. By the claim, $\mathcal{M}(P_{m_1,m_2}', P_{-m_1,m_2}) \subseteq \{\mu_1, \mu_3\}$. By the SP for m_2, $f(P_{m_1,m_2}', P_{-m_1,m_2}) = \mu_1$. Let $P_{w_2}' : m_1, m_2, m_3$.[12] Note that $(P_{m_2,w_2}', P_{-m_2,w_2}) \in \hat{\boldsymbol{P}}$. By the claim, $\mathcal{M}(P_{m_1,m_2,w_2}', P_{-m_1,m_2,w_2}) \subseteq \{\mu_1, \mu_3\}$. By the SP for w_2, $f(P_{m_1,m_2,w_2}', P_{-m_1,m_2,w_2}) = \mu_1$.

Let $P_{w_1}' : m_3, m_2, m_1$[13] and $\hat{P}_{w_1} : m_2, m_3, m_1$.[14] Note that $(P_{m_2,w_2}', \hat{P}_{w_1}, P_{-m_2,w_1,w_2}) \in \hat{\boldsymbol{P}}$. By the claim, $\mathcal{M}(P_{m_1,m_2,w_2}', \hat{P}_{w_1}, P_{m_3,w_3}) \subseteq \{\mu_1, \mu_3\}$. By the SP for w_1, $f(P_{m_1,m_2,w_2}', \hat{P}_{w_1}, P_{m_3,w_3}) =$

[9] Equivalently,

$$\begin{aligned}
\mu_1, \mu_2 P_{m_1} \mu_3, \mu_4 \ \& \ \mu_1, \mu_2 P_{m_1} \mu_5, \mu_6,\\
\mu_1, \mu_6 P_{m_2} \mu_3, \mu_5 \ \& \ \mu_1, \mu_6 P_{m_2} \mu_2, \mu_4,\\
\mu_1, \mu_3 P_{m_3} \mu_4, \mu_6 \ \& \ \mu_1, \mu_3 P_{m_3} \mu_2, \mu_5,\\
\mu_3, \mu_5 P_{w_1} \mu_1, \mu_2 \ \& \ \mu_3, \mu_5 P_{w_1} \mu_4, \mu_6,\\
\mu_3, \mu_4 P_{w_2} \mu_1, \mu_6 \ \& \ \mu_3, \mu_4 P_{w_2} \mu_2, \mu_5,\\
\mu_1, \mu_3 P_{w_3} \mu_5, \mu_6 \ \& \ \mu_1, \mu_3 P_{w_3} \mu_2, \mu_4.
\end{aligned}$$

[10] Equivalently, $R_{m_1}' : \mu_5, \mu_6; \mu_1, \mu_2; \mu_3, \mu_4$.
[11] Equivalently, $R_{m_2}' : \mu_1, \mu_6; \mu_2, \mu_4; \mu_3, \mu_5$.
[12] Equivalently, $R_{w_2}' : \mu_3, \mu_4; \mu_1, \mu_6; \mu_2, \mu_5$.
[13] Equivalently, $R_{w_1}' : \mu_4, \mu_6; \mu_3, \mu_5; \mu_1, \mu_2$
[14] Equivalently, $\hat{R}_{w_1} : \mu_3, \mu_5; \mu_4, \mu_6; \mu_1, \mu_2$.

μ_1. By the SP for w_1, $f(P'_{m_1,m_2,w_1,w_2}, P_{m_3,w_3})$ is either μ_1 or μ_2. Note that μ_1 beats μ_2 under $(P'_{m_1,m_2,w_1,w_2}, P_{m_3,w_3})$ since $\mu_1 P'_{m_2} \mu_2$, $\mu_1 P'_{w_2} \mu_2$, $\mu_1 P_{m_3} \mu_2$, and $\mu_1 P_{w_3} \mu_2$. Thus, $f(P'_{m_1,m_2,w_1,w_2}, P_{m_3,w_3}) = \mu_1$. Let $P'_{w_3} : m_3, m_1, m_2$.[15] By the SP for w_3, $f(P_{m_3}, P'_{-m_3})$ is either μ_1 or μ_3. Let $P'_{m_3} : w_2, w_3, w_1$.[16] By the SP for m_3, $f(P') \neq \mu_6$.

Note that P' corresponds to "P^{aBC}" in Lemma 1, considering that $a = m_1$, $b = m_3$, $c = m_2$, $A = w_3$, $B = w_2$, and $C = w_1$. As the preference profiles corresponding to "P^{aC}" and "P^{AC}," let P^1 and P^2 be

$$P^1_{w_2} : m_1, m_3, m_2, \quad \& \quad \forall i \neq w_2,\ P^1_i = P'_i$$

$$P^2_{m_1} : w_3, w_2, w_1, \quad P^2_{w_2} (= P^1_{w_2}) : m_1, m_3, m_2, \quad P^2_{w_3} : m_3, m_2, m_1,^{17} \ \& \ \forall i \notin \{m_1, w_2, w_3\}\ P^2_i = P'_i.$$

Since $f(P') \neq \mu_6$, by Lemma 1 (i), we have $f(P^1) = f(P^2) = \mu_3$. Let $P^*_{m_2} : w_3, w_1, w_2$.[18] Since $P^1_{m_2} = P^2_{m_2} = P'_{m_2}$, both $f(P^*_{m_2}, P^1_{-m_2})$ and $f(P^*_{m_2}, P^2_{-m_2})$ are not μ_4.

Now, we reconsider that $a = w_2$, $b = w_3$, $c = w_1$, $A = m_1$, $B = m_3$, and $C = m_2$ (note that Lemma 1 simultaneously holds for any correspondence relationship between $M \cup W$ and $\{a,b,c\} \cup \{A,B,C\}$). Then, $(P^*_{m_2}, P^1_{-m_2})$ and $(P^*_{m_2}, P^2_{-m_2})$ correspond, respectively, to "P^{cAC}" and "P^{bcC}" in Lemma 1. By Lemma 1 (ii), f is not SP or not majority, and this is a contradiction.

For the case where $f(P) = \mu_3$, we can obtain the similar proof by replacing all m_1, m_2, m_3, w_1, w_2, and w_3 in the former case with w_1, w_2, w_3, m_2, m_1, and m_3, respectively. $\square$

Proposition 1 shows that there are no majority and SP rules for the pure matching problem between three men and three women, and this result is valid when we consider the problems between k men and k women with $k \geq 3$. For example, consider the problems between four men m_1, m_2, m_3, and m_4, and four women w_1, w_2, w_3, and w_4. Now, there are 24 possible (pure) matchings. Considering the preference profiles in which (i) m_1, m_2, and m_3 prefer w_4 the worst, (ii) w_1, w_2, and w_3 prefer m_4 the worst, and (iii) m_4 and w_4 prefer each other the best, only possibility for m_4 and w_4 to match with each other occurs for the majority. In other words, there are substantially six possible matchings. This means that the approach used for the problems between three men and three women is also applicable to this case. Meanwhile, the following proposition demonstrates that for pure matching problems between two men and two women, there exists a stable (and thus, majority) rule and an SP rule.

Proposition 2. *If $\#M = \#W = 2$, then there exists a rule that is both stable and SP.*

Proof. For each $P \in \boldsymbol{P}$, let $f^M(P)$ be such that $f^M(P) \in \mathcal{S}(P)$, and $f^M(P) R_i \mu$ for any $i \in M$ and any stable matching μ. Note that Theorem 2 in Gale and Shapley (1962) shows that such a rule exists for any preference profile $P \in \boldsymbol{P}$. Note also that Theorem 5 in Roth (1982) proves that $f^M(P)$ is SP for men for any preference profile $P \in \boldsymbol{P}$, and thus, we only need to check the SP for women for the rule f^M.

[15] Equivalently, $R'_{w_3} : \mu_1, \mu_3; \mu_5, \mu_6; \mu_2, \mu_4$.

[16] Equivalently, $R'_{m_3} : \mu_2, \mu_5; \mu_1, \mu_3; \mu_4, \mu_6$.

[17] Equivalently, $R^1_{w_2} : \mu_3, \mu_4; \mu_2, \mu_5; \mu_1, \mu_6$, $R^2_{m_1} : \mu_5, \mu_6; \mu_3, \mu_4; \mu_1, \mu_2$, and $R^2_{w_3} : \mu_1, \mu_3; \mu_2, \mu_4; \mu_5, \mu_6$.

[18] Equivalently, $R^*_{m_2} : \mu_2, \mu_4; \mu_3, \mu_5; \mu_1, \mu_6$.

Given that we treat m_1 and m_2 symmetrically and also treat w_1 and w_2 symmetrically in f^M, and by the definition of $\boldsymbol{P}_i$ for $i \in M \cup W$, it is enough to show that $f^M(P)R_{w_1} f^M(P'_{w_1}, P_{-w_1})$ for each $P \in \boldsymbol{P}$ with $P_{w_1} : m_1, m_2$ and $P'_{w_1} : m_2, m_1$. This is trivial if w_1 matches with m_1 in $f^M(P)$. Thus, let w_1 match with m_2 in $f^M(P)$. By the definition of f^M, there exist only two cases: case (i) $P_{m_1} : w_2, w_1$, and $P_{m_2} : w_1, w_2$, and case (ii) $P_{m_1} : w_2, w_1$, $P_{m_2} : w_2, w_1$, and $P_{w_2} : m_1, m_2$. For each case, w_1 matches with m_2 in $f^M(P'_{w_1}, P_{-w_1})$. Thus, $f(P)R_{w_1} f(P'_{w_1}, P_{-w_1})$. □

The following proposition demonstrates that minimizing choice numbers and the SP are incompatible if there are three men and three women.

Proposition 3. *There is no f that minimizes choice numbers and is SP.*

Proof. Let $P \in \boldsymbol{P}$, $P'_{m_1} \in \boldsymbol{P}_{m_1}$, and $P'_{w_1} \in \boldsymbol{P}_{w_1}$ be such that

$$
\begin{aligned}
P_{m_1} &: w_1, w_2, w_3,\\
P_{m_2} &: w_2, w_3, w_1,\\
P_{m_3} &: w_3, w_1, w_2,\\
P_{w_1} &: m_3, m_1, m_2,\\
P_{w_2} &: m_1, m_2, m_3,\\
P_{w_3} &: m_2, m_3, m_1,\\
P'_{m_1} &: w_1, w_3, w_2,\\
P'_{w_1} &: m_3, m_2, m_1.
\end{aligned}
$$

Let f be such that it minimizes choice numbers and is SP. Then, $f(P'_{m_1}, P_{-m_1}) = \mu_1$ since it is the only matching minimizing choice numbers under (P'_{m_1}, P_{-m_1}). Since μ_1 and μ_4 are the only two matchings minimizing choice numbers under P, by the SP for m_1, we have $f(P) = \mu_1$. This contradicts the SP for w_1 since μ_4 is the only matching minimizing choice numbers under (P'_{w_1}, P_{-w_1}) and, thus, $f(P'_{w_1}, P_{-w_1}) = \mu_4$. □

Again, note that we can derive the same result for the case of $\#M = \#W > 3$. Meanwhile, the following proposition, together with Proposition 2, demonstrates that for pure matching problems between two men and two women, there exists a rule that is SP and minimizes choice numbers.

Proposition 4. *If $\#M = \#W = 2$, then every stable matching minimizes choice numbers.*

Proof. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Note that there are only two matchings: $\{m_1w_1, m_2w_2\}$ and $\{m_1w_2, m_2w_1\}$. Let $P \in \boldsymbol{P}$ and $\mu \in \mathcal{S}(P)$. Suppose that there exists $(m, w) \in M \times W$ with $wP_m w'$ and $mP_w m'$ for $w' \neq w$ and $m' \neq m$, respectively. Since $\mu \in \mathcal{S}(P)$, $\mu(m) = w$. Thus, for $\mu' \neq \mu$, $\mu'(m) = w'$ and $\mu'(w) = m'$. That is, $c(\mu(i), P_i) = 0$ and $c(\mu'(i), P_i) = 1$ for $i \in \{m, w\}$. Thus, $\sum_i c(\mu(i), P_i) \leq 2 \leq \sum_i c(\mu'(i), P_i)$. Then, μ refers to minimizing choice numbers.

Suppose that there exists no such $(m, w) \in M \times W$. There are only two (symmetric) cases: (i) $w_1P_{m_1}w_2$, $w_2P_{m_2}w_1$, $m_2P_{w_1}m_1$, and $m_1P_{w_2}m_2$, and (ii) $w_2P_{m_1}w_1$, $w_1P_{m_2}w_2$, $m_1P_{w_1}m_2$, and $m_2P_{w_2}m_1$. For each case, the sum of the choice numbers of $\{m_1w_1, m_2w_2\}$ is equal to the sum of those of $\{m_1w_2, m_2w_1\}$; the number is 2. Thus, μ refers to minimizing choice numbers. □

Acknowledgments: This study is supported by the JSPS KAKENHI Grant Numbers 25780139 (Adachi) and 15K17031 (Kongo).

References

Alcalde J, Barberà S (1994) Top dominance and the possibility of strategy-proof stable solutions to matching problems. Economic Theory 4:417–435

Gale D, Shapley LS (1962) College admissions and the stability of marriage. American Mathematical Monthly 69:9–15

Gärdenfors P (1975) Match making: assignments based on bilateral preferences. Behavioral Science 20:166–173

Gibbard A (1973) Manipulation of voting schemes: A general results. Econometrica 41:587–601

Irving RW, Leather P, Gusfield D (1987) An efficient algorothm for the "optimal" stable marriage. Journal of the Assosiation for Computing Machinery 34:532–543

Kongo T (2013) An incompatibility between recursive unanimity and strategy-proofness in two-sided matching problems. Social Choice and Welfare 40:461–478

McVitie DG, Wilson LB (1971) The stable marriage problem. Communications of the ACM 14:486–490

Roth AE (1982) The economics of matching: Stability and incentives. Mathematics of Operations Research 7:617–628

Roth AE (1991) A natural experiment in the organization of entry level labor markets: Regional markets for new physicians and surgeons in the United Kingdom. American Economic Review 81:415–440

Roth AE (2002) The economist as engineer: Game theory, experimentation, and computation as tools for design economics. Econometrica 70:1341–1378

Satterthwaite MA (1975) Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory 10:187–217

Sönmez T (1996) Strategy-proofness in many-to-one matching problems. Economic Design 1:365–380

Tadenuma K, Toda M (1998) Implementable stable solutions to pure matching problems. Mathematical Social Sciences 35:121–132

Takagi S, Serizawa S (2010) An impossibility theorem for matching problems. Social Choice and Welfare 35:245–266

Toda M (2006) Monotonicity and consistency in matching markets. International Journal of Game Theory 34:13–31