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Marginalism and egalitarianism under the equal effect of players' nullification

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Abstract

We provide two axiomatic characterizations of the solutions of TU games on the fixed player set, where at least three players exist. We introduce three axioms on players' nullification. One axiom requires that the difference between the effect of a player's nullification on the nullified player and that on the other is relatively constant in any situation and given whoever is nullified. This axiom characterizes the equal division value together with the two basic axioms of null game and efficiency. Further, two weaker variations of the first axiom characterize the class of linear combinations of the equal surplus division and equal division values, together with the two basic axioms.

Keywords: TU game, axiomatization, equal division value, equal surplus division value, nullification

JEL Classification: C71

1 Introduction

Some economic problems regarding allocating agents' cooperative outcomes are adequately described by cooperative games with transferable utility (TU games). In TU games, such allocations are represented as solutions, of which a wide variety have been proposed in the literature. These solutions can be applied to many economic situations such as cost sharing in the construction of airport runways (Littlechild and Owen 1973 and Littlechild 1974) and in cooperative water resource development (Suzuki and Nakayama 1976), sharing the revenue from museum passes (Ginsburgh and Zang 2003 and Bergantiños and Moreno-Ternero 2015), and the patent licensing of a cost-reducing technology in a Cournot market (Tauman and Watanabe 2007).

One well-known one-point solution for TU games (also known as the value of games) that determines a single allocation for each game is the Shapley value (Shapley 1953). In this value, we focus on how each player's existence affects the worth of coalitions (i.e., each player's marginal contribution to the coalition).

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Any player whose elimination from the coalition does not change its worth (a so-called null player) obtains nothing. In this sense, the Shapley value exemplifies marginalism in TU games.[1]

Another well-known value of games located at the other end to the Shapley value is the equal division value. This value divides the worth obtained by all players' grand coalition equally among them. Thus, it never serves players differently even if their marginal contributions are different. In this sense, the equal division value exemplifies egalitarianism in TU games.

Some values of games reconcile marginalism with egalitarianism in TU games in several manners. For example, the equal surplus division value (also known as the CIS value, Driessen and Funaki 1991)[2] first allocates each player his or her marginal contribution to the singleton coalition of himself or herself, and then allocates each player the equal division of the remainder (i.e., the equal division of the worth of the grand coalition minus the sum of the worth of every player's standalone coalition). Hence, this value partly respects marginalism and partly respects egalitarianism in TU games. A similar observation holds for the equal allocation of the non-separable contribution value (also known as the ENSC value, Driessen and Funaki 1991),[3] which first allocates each player his or her marginal contribution to the grand coalition and then allocates each player the equal division of the remainder.[4]

Further, some mixtures of a marginalistic value and an egalitarian value (values) are also considered as convex combinations of them. Such values well-known in the literature include α-egalitarian Shapley values (Joosten 1996, van den Brink et al. 2013; convex combinations of the Shapley and equal division values), consensus values (Ju et al. 2007; convex combinations of the Shapley and equal surplus division values), equal surplus solutions (van den Brink and Funaki 2009; convex combinations of the equal division, equal surplus division, and ENSC values), and convex combinations of the α-egalitarian Shapley and consensus values (Yokote and Funaki 2015).

In this paper, we give two axiomatizations of solutions of games on the class of all TU games on a fixed player set, which contains more than or equal to three players. One is an axiomatization of the equal division value. The other is an axiomatization of the class of all linear combinations of the equal division and equal surplus division values. Characterizations of either the equal division value or the equal surplus division value and their convex combinations are (particularly recently) well-studied by Béal et al. (2014a, 2016a), van den Brink (2007, 2009), van den Brink et al. (2012), van den Brink and Funaki (2009), Casajus and Huettner (2014), Chun and Park (2012), and Kamijo and Kongo (2012, 2015), and this paper follows in the wake of these previous studies. Both characterizations in this paper incorporate the two basic axioms of null game and efficiency,[5] and axiom(s) on the effects of players' nullification on allocations

[1] The axiomatizations of the Shapley value by Young (1985) and Casajus (2011) capture this aspect of the value well.

[2] CIS stands for the center of gravity of the imputation set.

[3] ENSC stands for egalitarian non-separable contribution.

[4] Additionally, the solidarity value (Nowak and Radzik 1994), which considers the average contributions of all players in a coalition instead of a player's marginal contribution to a coalition in the definition of the Shapley value, can also be seen as a mixture of marginalism and egalitarianism in TU games.

[5] The axiom of null game requires that every player receives zero if games assign zero to every coalition, and efficiency requires that the sum of the allocations of all players is equal

of players.

A concept of players' nullification investigates the effects of a player becoming a null player in a game on the allocations of players. Recently, Béal et al. (2016b) point out that this concept is related to collusion properties by proxy agreements studied in Haller (1994) and that, in various economic situations, players' nullification makes sense. They formulate three axioms related to players' nullification and each of their axioms corresponds to players' nullification versions of the balanced contributions property of Myerson (1980),[6] the balanced collective contributions property of Béal et al. (2014b), and the balanced cycle contributions property of Kamijo and Kongo (2010). Their axioms focus on the effects of a player's nullification on the allocations of a non-nullified player(s). Meanwhile, our axioms focus on the comparison of the effects of a player's nullification on allocations between nullified and non-nullified players. Players' nullification varies allocations not only for non-nullified players but also for the nullified player, meaning that differences in the effects between the nullified and other players are also worth investigating.

Our first axiom related to players' nullification requires that the difference between the effect of a player's nullification on allocations for the nullified player and that for each of the other players is constant in any situation and based on whoever becomes nullified. More precisely, we compare the relative effects to the change in the worth of the grand coalition because the number of the changes of allocations are supposed to be larger as the change in the worth of the grand coalition rises (note that we assume efficiency as the basic axiom). We call the axiom the equal effects of players' nullification. This axiom can be interpreted by the following two requirements: (i) we treat all situations of a player's nullification in the same manner, and (ii) in each game, we treat all players other than the nullified in the same manner. In this sense, our first axiom is a kind of fairness condition on players' nullification. This axiom, together with the above two basic axioms, characterizes the equal division value.

Our second and third axioms related to players' nullification are weaker variations of the first axiom, and they broaden the characterized solutions. The second axiom, which is called the equal effects of players' nullification for almost null games, requires the same condition as the first one only in so-called almost null games. Almost null games are TU games in which only one player affects the worth of the coalition (i.e., all but one players are null players). This axiom weakens requirement (i) of the first axiom, and together with the basic axiom of the null game, it respects the linear combinations of marginalism and egalitarianism for almost null games.

The third axiom, which is called the equal effect of players' nullification on the others, needs only requirement (ii) of the first axiom. This axiom requires that the effects of a player's nullification are the same for any non-nullified player.[7] This is motivated by the fact that non-nullified players are equally free of responsibility to the change in the amount shared by all players, and hence the fair treatment of the players requires that all non-nullified players should be affected equally. These two weaker axioms, together with the above

to the worth of the grand coalition.

[6]The same property is also studied by Gomez-Rua and Vidal-Puga (2010).

[7]Note that, independently of this paper, Ferrières (2016) also studies a property on the effects of a player's nullification on allocations between non-nullified players, which is equivalent to our third axiom related to players' nullification.

two basic axioms, characterize the class of solutions represented by all linear combinations between the equal division and equal surplus division values. By comparing our two characterizations, the difference between our first axiom and the combination of the second and third axioms allows us to partly respect marginalism in solutions for TU games.

The remainder of this paper is structured as follows. Section 2 is for the preliminaries. Section 3 introduces the three new axioms related to players' nullification. Section 4 provides our main results on the axiomatization of the equal division value and that of the class of all linear combinations of the equal division and equal surplus division values.

2 Preliminaries

Let $N \subseteq \mathbb{N}$ be a **set of players**. Throughout this paper, we fix N. In addition, we assume that $\#N = n \geq 3$, where $\#$ denotes the cardinality of a set. A singleton subset $\{i\} \subseteq N$ is written by i if it leaves no room for confusion. A characteristic function $v : 2^N \to \mathbb{R}$, where $v(\emptyset) = 0$, is called a **TU game** on N. A set of all TU games on N is denoted by $\mathbb{V}(N)$. Given a game v, a player $i \in N$, and a subset $S \subseteq N$ with $S \ni i$, $v(S) - v(S \setminus i)$ is called **i's marginal contribution to S**.

Given a game $v \in \mathbb{V}(N)$, a player $i \in N$ satisfying $v(S) = v(S \setminus i)$ for any $S \subseteq N$ with $S \ni i$ is called a **null player** in v. Let $NP(v) \subseteq N$ be a set of null players in a game v. A TU game $\mathbf{0} \in \mathbb{V}(N)$, where $\mathbf{0}(S) = 0$ for any $S \subseteq N$, is referred to as a **null game**. By using the term null players, the null game can be seen as a game in which all players are null players (i.e., $NP(\mathbf{0}) = N$). A TU game $v \in \mathbb{V}(N)$ is called an **almost null game** if in the game, all players except one are null players (i.e., if there exists a player $i \in N$ such that $\{i\} = N \setminus NP(v)$).

Next, we consider a player's nullification in games. Given a game $v \in \mathbb{V}(N)$, a **player $i \in N$ is nullified in a game** $v^i \in \mathbb{V}(N)$ if $v^i(S) = v(S \setminus i)$ for any $S \subseteq N$. Put simply, a player $i \in N$ becomes a null player in the nullified game $v^i \in \mathbb{V}(N)$ (i.e., $NP(v^i) = NP(v) \cup i$).

Let $\varphi : \mathbb{V}(N) \to \mathbb{R}^N$ be a one-point solution or a **value** for games. Each coordinate of a value represents the payoff of a player in the game. Three well-known values of games are the Shapley value (Shapley 1953), the equal division value, and the equal surplus division value (Driessen and Funaki 1991).

The **Shapley value** $Sh : \mathbb{V}(N) \to \mathbb{R}^N$ is the expected value of a player's marginal contribution to any coalition that contains the player. Hence, it is a representative example of a marginalistic value and is defined as follows: for any $v \in \mathbb{V}(N)$ and any $i \in N$,

$$Sh_i(v) = \sum_{S \ni i} \frac{(\#S - 1)!(n - \#S!)}{n!}(v(S) - v(S \setminus i)).$$

The **equal division value** $ED : \mathbb{V}(N) \to \mathbb{R}^N$ is the simple equal division of the worth of the grand coalition N. Thus, it is a representative example of an egalitarian value and is defined as follows: for any $v \in \mathbb{V}(N)$ and any $i \in N$,

$$ED_i(v) = \frac{v(N)}{n}.$$

The **equal surplus division value** $ESD : \mathbb{V}(N) \to \mathbb{R}^N$ is the worth of the standalone coalition of each player plus an equal division of the surplus obtained by the grand coalition, which is the difference between the worth of the grand coalition and the sum of the worth of each player's standalone coalition. It is defined as follows: for any $v \in \mathbb{V}(N)$ and any $i \in N$,

$$ESD_i(v) = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}.$$

In the following, we introduce the two basic axioms on values.

Null Game (NG): For any $i \in N$, $\varphi_i(\mathbf{0}) = 0$.

Efficiency (EF): For any $v \in \mathbb{V}(N)$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Because these two are well known in the literature, we omit detailed discussions on them.

3 Effects of players' nullification on allocations

In this section, we introduce three axioms on the effects of players' nullification on the allocations of players. One axiom mainly compares these effects between nullified and other players. The other two axioms are weaker than the first one. One weaker axiom requires the same condition as the first one only to specific games. The other weaker axiom compares the effects only between players other than the nullified player. All three axioms are meaningful only when there are at least three players in games, and thus we restrict our attention to games that have more than three players.

We explain our first axiom related to the effects of players' nullification on players. Suppose that $i \in N$ is **not** a null player in the original game $v \in \mathbb{V}(N)$. Player i's nullification changes the worth of the grand coalition from $v(N)$ to $v(N \setminus i)$. Accordingly, the allocations of the players vary. Because these changes are attributed to only player i, the amount of the change for a player i, $\varphi_i(v) - \varphi_i(v^i)$, and that for the other player $k \neq i$, $\varphi_k(v) - \varphi_k(v^i)$, are supposed to vary. In addition, the amounts of the changes for players other than i should be the same because all of them are equally free of responsibility for the changes in the worth of the coalitions containing i. Furthermore, those changes seem to rise as the changes in the worth of the grand coalition, $v(N) - v(N \setminus i)$, increases. Based on these observations, the difference in the effects of a player i's nullification on players between the nullified and another player $k \neq i$ is appropriately evaluated by $\frac{(\varphi_i(v) - \varphi_i(v^i)) - (\varphi_k(v) - \varphi_k(v^i))}{v(N) - v(N \setminus i)}$. Our first axiom requires that this value is constant from any situation and for any player who becomes nullified.

Equal Effects of Players' Nullification (EEN): For any game $v, w \in \mathbb{V}(N)$, any $i, j \in N$ such that $i \in N \setminus NP(v)$ and $j \in N \setminus NP(w)$, and any $k \neq i, j$,

$$\frac{(\varphi_i(v) - \varphi_i(v^i)) - (\varphi_k(v) - \varphi_k(v^i))}{v(N) - v(N \setminus i)} = \frac{(\varphi_j(w) - \varphi_j(w^j)) - (\varphi_k(w) - \varphi_k(w^j))}{w(N) - w(N \setminus j)}.$$

In other words, throughout society, there are agreed conditions on the symmetric treatment of all players with respect to becoming null players and that of all non-nullified players with respect to their allocations. This specific conditions differ from one value of games to another. Note that this axiom does not exclude the same treatment between nullified and other players, and thus it is satisfied by ED.

The next axiom requires the same condition as EEN; however, it applies the condition for almost null games only.

EEN for Almost Null Game (EENA): For any almost null game $v, w \in \mathbb{V}(N)$, any $i, j \in N$ such that $i \in N \setminus NP(v)$ and $j \in N \setminus NP(w)$, and any $k \neq i, j$,

$$\frac{(\varphi_i(v) - \varphi_i(v^i)) - (\varphi_k(v) - \varphi_k(v^i))}{v(i)} = \frac{(\varphi_j(w) - \varphi_j(w^j)) - (\varphi_k(w) - \varphi_k(w^j))}{w(j)}.$$

Note that since v and w are almost null games, $v(i) = v(N) - v(N \setminus i)$ and $w(j) = w(N) - w(N \setminus j)$. Obviously EENA is weaker than EEN.

The last axiom requires the symmetric treatment of all non-nullified players only. Hence, we can treat all players asymmetrically with respect to who becomes a null player.

Equal Effect of Players' Nullification on Others (EENO): For any game $v \in \mathbb{V}(N)$, any $i \in N$, and any $j, k \neq i$, $\varphi_j(v) - \varphi_j(v^i) = \varphi_k(v) - \varphi_k(v^i)$.

EENO is also weaker than EEN and it is independent of EENA (for details, see the independence of axioms in Theorem 2 in Section 4).

4 Axiomatizations

We first introduce two lemmas.

Lemma 1. *If $\varphi : \mathbb{V}(N) \to \mathbb{R}^N$ satisfies NG and EENA (or EEN), then there exists a common $\alpha \in \mathbb{R}$ such that for any almost null game $v \in \mathbb{V}(N)$, where $\{i\} = N \setminus NP(v)$, and for any $k \neq i$, it holds that $\varphi_i(v) - \varphi_k(v) = \alpha v(i)$.*

Proof. Let $v, w, u \in \mathbb{V}(N)$ be almost null games, where $\{i\} = N \setminus NP(v)$, $\{j\} = N \setminus NP(w)$, $\{h\} = N \setminus NP(u)$, $i \neq j$, $j \neq h$, and $h \neq i$. Now, $v(N) - v(N \setminus i) = v(i)$, $v^i = \mathbf{0}$, $w(N) - w(N \setminus j) = w(j)$, $w^j = \mathbf{0}$, $u(N) - u(N \setminus h) = u(h)$, and $u^h = \mathbf{0}$. By NG, $\varphi_\ell(v^i) = \varphi_\ell(w^j) = \varphi_\ell(u^h) = 0$ for any $\ell \in N$. Hence, by EENA (or EEN), we obtain

$$\frac{\varphi_i(v) - \varphi_k(v)}{v(i)} = \frac{\varphi_j(w) - \varphi_k(w)}{w(j)} \text{ for } k \neq i, j, \text{ and}$$
$$\frac{\varphi_i(v) - \varphi_k(v)}{v(i)} = \frac{\varphi_h(u) - \varphi_k(u)}{u(h)} \text{ for } k \neq i, h.$$

By letting $\frac{\varphi_j(w) - \varphi_k(w)}{w(j)} = \frac{\varphi_h(u) - \varphi_k(u)}{u(h)} = \alpha$, we have the desired result. □

In almost null games, a non-null player i's marginal contribution to any subset $S \ni i$ is $v(i)$ and any other player k's marginal contributions to any subset $S \ni k$ is 0. Hence, if $\alpha = 1$, NG and EENA (or EEN) together require that the

difference between the allocation of a non-null player and that of a null player is equivalent to the difference in their marginal contributions; in other words, the two axioms respect marginalism in almost null games. Meanwhile, if $\alpha = 0$, the two axioms respect egalitarianism in almost null games. Therefore, the two axioms together imply some linear combination between marginalism and egalitarianism in almost null games. Further, together with EF, the following lemma clarifies the fact from the standpoint of the values of games.

Lemma 2. *If $\varphi : \mathbb{V}(N) \to \mathbb{R}^N$ satisfies NG, EF, and EENA (or EEN), then there exists a common $\alpha \in \mathbb{R}$ such that for any almost null game $v \in \mathbb{V}(N)$, where $\{i\} = N \setminus NP(v)$, it holds that $\varphi(v) = \alpha ESD(v) + (1-\alpha)ED(v)$.*

Proof. Let $v \in \mathbb{V}(N)$ be an almost null game, where $\{i\} = N \setminus NP(v)$. Now $v(N) = v(i)$. According to EF and Lemma 1, $v(i) = \sum_{j \in N} \varphi_j(v) = n\varphi_k(v) + \alpha v(i)$, for any $k \neq i$. Thus, $\varphi_k(v) = (1-\alpha)\frac{v(i)}{n} = \alpha ESD_k(v) + (1-\alpha)ED_k(v)$, for any $k \neq i$, and $\varphi_i(v) = \alpha v(i) + (1-\alpha)\frac{v(i)}{n} = \alpha ESD_i(v) + (1-\alpha)ED_i(v)$. □

Lemma 2 is used for the characterizations of ED (Theorem 1) and the class of linear combinations of ESD and ED (Theorem 2) below.

Theorem 1. *ED is characterized by NG, EF, and EEN.*

Proof. Let $\varphi : \mathbb{V}(N) \to \mathbb{R}^N$ satisfy the three axioms. We prove the fact by an induction with respect to the number of non-null players in $v \in \mathbb{V}(N)$.

First, we consider a game $v \in \mathbb{V}(N)$, where $|N \setminus NP(v)| = 0$, (v is the null game). By NG, $\varphi(v) = ED(v)$. Next, we consider a game $v \in \mathbb{V}(N)$, where $|N \setminus NP(v)| = 1$, (v is an almost null game). From Lemma 2, φ is represented by the linear combination of ESD and ED for some common $\alpha \in \mathbb{R}$.

In accord with this fact, we consider a game $w \in \mathbb{V}(N)$, where $|N \setminus NP(w)| = 2$. Let $i, j \in N \setminus NP(w)$ be two distinct players. Note that $w(N) = w(\{i,j\})$, $w(N \setminus i) = w(j)$, and $w(N \setminus j) = w(i)$. Note also that w^i and w^j are almost null games. Hence, according to Lemma 1, $\varphi_i(w^i) = \varphi_k(w^i)$ for any $k \neq j$, and $\varphi_j(w^j) = \varphi_k(w^j)$ for any $k \neq i$.

Let $h \neq i$, $h \neq j$, and $u \in \mathbb{V}(N)$, where $\{h\} = N \setminus NP(u)$. Note that u^h is the null game, and hence by NG, $\varphi_\ell(u^h) = 0$ for any $\ell \in N$. Since u is an almost null game, Lemma 1 implies that $\frac{\varphi_h(u) - \varphi_k(u)}{u(h)} = \alpha$, for any $k \neq h$.

Now, from EEN

$$\frac{\varphi_i(w) - \varphi_k(w)}{w(\{i,j\}) - w(j)} = \frac{\varphi_h(u) - \varphi_k(u)}{u(h)} = \alpha \text{ for } k \neq i, h, \text{ and} \tag{1}$$

$$\frac{\varphi_j(w) - \varphi_k(w)}{w(\{i,j\}) - w(i)} = \frac{\varphi_h(u) - \varphi_k(u)}{u(h)} = \alpha \text{ for } k \neq j, h. \tag{2}$$

Eq.(1) is equivalent to $\varphi_i(w) - \varphi_j(w) = \alpha w(\{i,j\}) - \alpha w(j)$, if $k = j$, and eq.(2) is equivalent to $\varphi_j(w) - \varphi_i(w) = \alpha w(\{i,j\}) - \alpha w(i)$, if $k = i$. The two equations together imply that $\alpha(2w(\{i,j\}) - w(i) - w(j)) = 0$. This holds for any game w, where $\{i,j\} = N \setminus NP(w)$. Hence, we get $\alpha = 0$ and thus, for any almost null game v, $\varphi(v) = ED(v)$. Further, given that $\alpha = 0$, according to eqs.(1) and (2) as well as EF, it holds that $\varphi(w) = ED(w)$, such that $|N \setminus NP(w)| = 2$.

Assume that for any $v \in \mathbb{V}(N)$ with $|N \setminus NP(v)| = m \geq 2$, it holds that $\varphi(v) = ED(v)$. Consider any $w \in \mathbb{V}(N)$ with $|N \setminus NP(w)| = m + 1$. Let $i, j \in$

$N \setminus NP(w)$ be two distinct players. From EEN and an induction hypothesis, we obtain $\varphi_i(w) = \varphi_j(w) = \varphi_k(w)$, for any $k \neq i, j$. From EF, it holds that $\varphi(w) = ED(w)$. This completes the proof. □

In what follows, we confirm the independence of the axioms in Theorem 1.

- The Shapley value satisfies NG and EF, but not EEN.
- The value $\varphi^1(v) = \frac{ED(v)}{2}$ satisfies NG and EEN, but not EF.
- The value $\varphi_i^2(v) = ED(v) + t_i$, where $\sum_{i \in N} t_i = 0$, for any $i \in N$ satisfies EF and EEN, but not NG.

We compare the above characterization of ED with that of the Shapley value in Béal et al. (2016b). Béal et al. (2016b, Proposition 5) characterize the Shapley value by NG, EF, and the balanced contributions under nullification (BCN), which requires that for any pair of players, the effects of a player becoming a null player on the allocation of the other player are equal. By comparing the results, we see differences between the equal division value and the Shapley value, (i.e., between egalitarianism and marginalism in TU games), depicted as the differences between EEN and BCN.

Theorem 1 tells us that EEN together with the two basic axioms unifies the value of games. The following theorem shows that by replacing EEN with its two weaker axioms of EENO and EENA, the possibility of the values of games is enlarged to the class of values represented by all linear combinations of ESD and ED.

Theorem 2. *The class of values* $\{\beta ESD + (1-\beta)ED | \beta \in \mathbb{R}\}$ *is characterized by NG, EF, EENO, and EENA.*

Proof. Let A be a class of values satisfying the four axioms. We prove that (i) $\{\beta ESD + (1-\beta)ED | \beta \in \mathbb{R}\} \subseteq A$ and (ii) $A \subseteq \{\beta ESD + (1-\beta)ED | \beta \in \mathbb{R}\}$.

(i) Note that both ESD and ED satisfy the four axioms. Hence, any value $\varphi \in \{\beta ESD + (1-\beta)ED | \beta \in \mathbb{R}\}$ also satisfies the four axioms.

(ii) Let $\varphi \in A$. We prove that for any $v \in \mathbb{V}(N)$, $\varphi(v) \in \{\beta ESD(v) + (1-\beta)ED(v) | \beta \in \mathbb{R}\}$. The proof is obtained by an induction with respect to the number of non-null players in games.

First, we consider a game v, where $|N \setminus NP(v)| = 0$ (the null game). According to NG, $\varphi(v) = (0, 0, \ldots, 0) \in \{\beta ESD(v) + (1-\beta)ED(v) | \beta \in \mathbb{R}\}$.

Next, we consider a game v, where $|N \setminus NP(v)| = 1$ (an almost null game). From Lemma 2, $\varphi(v) \in \{\beta ESD(v) + (1-\beta)ED(v) | \beta \in \mathbb{R}\}$.

Next, we consider a game v, where $|N \setminus NP(v)| = 2$. Let i, j be two distinct players in $N \setminus NP(v)$, and fix $k \neq i, j$.[8] According to EENO,

$$\varphi_i(v) - \varphi_i(v^j) = \varphi_k(v) - \varphi_k(v^j), \tag{3}$$

$$\varphi_j(v) - \varphi_j(v^i) = \varphi_k(v) - \varphi_k(v^i), \tag{4}$$

and for any $\ell \neq i, j, k$,[9]

$$\varphi_\ell(v) - \varphi_\ell(v^i) = \varphi_k(v) - \varphi_k(v^i). \tag{5}$$

[8] Note that $|N \setminus NP(v)| \geq 2$, and by definition, $n \geq 3$.

[9] If this exists; otherwise (i.e., $N = \{i, j, k\}$), eqs. (3) and (4) are sufficient to lead to eq. (9).

Note that v^i and v^j are almost null games. From Lemma 1 and the facts that $v^j(i) = v(i)$, $v^i(j) = v(j)$, and $v^j(h) = v^i(h) = v(h) = 0$, for any $h \neq i, j$, there exists a common $\alpha \in \mathbb{R}$ such that

$$(3) \iff \varphi_i(v) = \varphi_k(v) - \alpha v(k) + \alpha v(i), \tag{6}$$

$$(4) \iff \varphi_j(v) = \varphi_k(v) - \alpha v(k) + \alpha v(j), \text{ and} \tag{7}$$

$$(5) \iff \varphi_\ell(v) = \varphi_k(v) - \alpha v(k) + \alpha v(\ell). \tag{8}$$

By summing (6), (7), and (8) for any $\ell \neq i, j, k$, we obtain the following:

$$\begin{aligned} &\sum_{h \neq k} \varphi_h(v) = (n-1)\varphi_k(v) - (n-1)\alpha v(k) + \sum_{h \neq k} \alpha v(h) \\ \iff &\sum_{h \neq k} \varphi_h(v) = (n-1)\varphi_k(v) - n\alpha v(k) + \sum_{h \in N} \alpha v(h). \end{aligned} \tag{9}$$

From EF, $\sum_{h \neq k} \varphi_h(v) = v(N) - \varphi_k(v)$, and thus

$$(9) \iff n\varphi_k(v) = v(N) + n\alpha v(k) - \sum_{h \in N} \alpha v(h) \iff \varphi_k(v) = \alpha ESD_k(v) + (1-\alpha) ED_k(v).$$

The above discussion is applicable for any $k \neq i, j$. Further, from the above fact,

$$\begin{aligned} (6) &\iff \varphi_i(v) = \alpha ESD_i(v) + (1-\alpha) ED_i(v), \text{ and} \\ (7) &\iff \varphi_j(v) = \alpha ESD_j(v) + (1-\alpha) ED_j(v). \end{aligned}$$

Therefore, $\varphi(v) \in \{\beta ESD(v) + (1-\beta) ED(v) | \beta \in \mathbb{R}\}$. Note that in a value $\varphi \in A$, the coefficient $\alpha \in \mathbb{R}$ is common to any game $v \in \mathbb{V}(N)$, where $|N \setminus NP(v)| = 2$, because α is common to any game $w \in \mathbb{V}(N)$, where $|N \setminus NP(w)| = 1$.

Let $m \geq 2$ and assume that $\varphi(w) \in \{\beta ESD(w) + (1-\beta) ED(w) | \beta \in \mathbb{R}\}$ for any $w \in \mathbb{V}(N)$, where $|N \setminus NP(w)| = m$. We consider a game v with $|N \setminus NP(v)| = m + 1$. The discussion is almost the same as in the above case of $|N \setminus NP(v)| = 2$. Hence, we omit it. This completes the proof. □

In what follows, we confirm the independence of the axioms in Theorem 2.

- The value $\varphi_i^3(v) = \frac{v(N) - \sum_{i \in N} \{v(i)\}^2}{n} + \{v(i)\}^2$ for any $i \in N$, satisfies NG, EF, and EENO, but not EENA. Hence, $\varphi^3 \notin \{\beta ESD + (1-\beta) ED | \beta \in \mathbb{R}\}$.
- The Shapley value Sh satisfies NG, EF, and EENA, but not EENO. Hence, $Sh \notin \{\beta ESD + (1-\beta) ED | \beta \in \mathbb{R}\}$.
- The value $\varphi^1(v) = \frac{ED(v)}{2}$ satisfies NG, EENO, and EENA, but not EF. Hence, $\varphi^1 \notin \{\beta ESD + (1-\beta) ED | \beta \in \mathbb{R}\}$.
- The value $\varphi_i^2(v) = ED(v) + t_i$, where $\sum_{i \in N} t_i = 0$, for any $i \in N$ satisfies EF, EENO, and EENA, but not NG. Hence, $\varphi^2 \notin \{\beta ESD + (1-\beta) ED | \beta \in \mathbb{R}\}$.

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