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Abstract

Axioms in TU cooperative games are classified into punctual and relational axioms. In view of treating symmetric players fairly, only punctual axioms have been discussed in the literature. This paper introduces a new relational axiom that describes the fair treatment of symmetric players. Our new axiom, the balanced contributions property for symmetric players, restricts the requirement of the balanced contributions property (Myerson (1980)) to two symmetric players. We first prove that even under efficiency, our new axiom is logically independent of *symmetry*, which requires that two symmetric players receive the same payoff. We next prove that in previous axiomatizations of an anonymous solution, replacing symmetry with our new axiom results in new axiomatizations of the solution. Our results strengthen existing solutions as fair allocation rules from the viewpoint of symmetric players.

Keywords: Game theory; Balanced contributions; Symmetry; TU cooperative games; Axiomatization

1 Introduction

One of the main targets in TU cooperative game theory is how to divide the cooperative surplus among players fairly. We describe an allocation problem by using a TU game (N, v), where N represents the set of players and v represents the attainable payoff for each coalition. Since v is the only information we can use, it is convincing to treat

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two symmetric players in v fairly, i.e., two players who affect the attainable payoff for each coalition in the same way. This idea of fairness is formulated as an axiom called *symmetry*, which requires that two symmetric players receive the same payoff.

Thomson (2012) introduced a categorization of axioms into punctual and relational axioms thus: "A punctual axiom applies to each problem separately and a relational axiom relates choices made across problems that are related in certain ways." Symmetry is a representative punctual axiom. On the contrary, Thomson's (2012) categorization also suggests the possibility of formulating a relational axiom that states the fair treatment of symmetric players. This paper introduces such an axiom and applies it to axiomatizations in an attempt to clarify the consequences of treating symmetric players fairly.

An essential part of a relational axiom is describing the change in a game situation. Recent studies focus on the change in the set of essential players caused by the departure of a player (see Kamijo and Kongo (2012), Casajus and Huettner (2014)) or the collusion of players (see Park and Ju (2016)). Following this line of the literature, we focus on a "reduced game" that a player leaves and examine the resulting effect on players' payoffs. Consider a game (N, v) and two symmetric players i and j. If player i leaves the game, then player i's departure has some effect on player j's payoff. Let us call the effect *player i's contribution to j's payoff.* Analogously, we can define player j's contribution to i's payoff. Now, i and j are symmetric, and hence, a fair relational axiom can require that these effects are the same for both players. This is our new axiom, termed the balanced contributions property for symmetric players (BCS hereafter).

We emphasize that the above idea of the balance of contributions dates back to Myerson's (1980) balanced contributions property. To clarify the relationship among the axioms discussed here, it is useful to examine the basic structure of a fairness criterion, which is formulated as an answer to the following question: *who should be treated how?* Symmetry treats two symmetric players in a way that their final payoffs are the same. Myerson's (1980) balanced contributions property treats *all players* in a way that their pairwise contributions are balanced. The comparison of these two axioms uncovers a drawback of Myerson's (1980) balanced contributions property. As a fairness criterion, its answer to the question "who" is too extreme. Our new axiom, by contrast, answers this question in a parallel manner with symmetry, i.e., BCS treats *two symmetric players* in a way that their contributions are balanced.

Our first result shows that, even under efficiency, symmetry and BCS are logically independent. In particular, BCS itself does not rule out the case in which two symmetric players receive different payoffs. This result poses the following question: in previous axiomatizations of solutions, if we replace symmetry with BCS, what kind of solution is characterized? Our second result answers this question. We prove that in previous axiomatizations of an anonymous solution, replacing symmetry with BCS results in new axiomatizations of the solution. In other words, in the context of axiomatizations, the combination of axioms bridges the gap between the punctual and relational axioms con-

cerning fairness among symmetric players. Our results thus strengthen existing solutions as fair allocation rules from the viewpoint of symmetric players.

The remainder of the paper is organized as follows. In Section 2, we introduce some preliminaries and define our new axiom, BCS. In Section 3, we discuss the logical relationship between BCS and symmetry. In Section 4, we apply BCS to axiomatizations.

2 Preliminary

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{N}$ denote the set of natural numbers, with the convention that $0 \notin \mathbb{N}$. In our analysis, we interpret $\mathbb{N}$ as representing the infinite universe of potential players. Let $\mathcal{N}$ denote the set of non-empty and finite subsets of $\mathbb{N}$. For $N \in \mathcal{N}$, let n denote $|N|$, the cardinality of N. A TU game is a pair (N, v), where $N \in \mathcal{N}$ and v is a characteristic function from 2^N to $\mathbb{R}$ satisfying $v(\emptyset) = 0$. Let Γ denote the set of all games. For $(N, v) \in \Gamma$ and $S \subseteq N$, $S \neq \emptyset$, with a slight abuse of the notation, let (S, v) denote the game in which the domain of v is restricted from 2^N to 2^S.

A solution is the function ψ that assigns a payoff vector $\psi(N, v) \in \mathbb{R}^N$ to each game $(N, v) \in \Gamma$. The Shapley value (Shapley (1953)) is defined by[1]

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{|S|!(n - |S| - 1)!}{n!} \big[v(S \cup i) - v(S)\big] \text{ for all } (N, v) \in \Gamma, i \in N.$$

The equal division value is defined by

$$ED_i(N, v) = \frac{v(N)}{n} \text{ for all } (N, v) \in \Gamma, i \in N.$$

For each game $(N, v) \in \Gamma$, let $\sim_{(N,v)}$ denote the binary relation on N defined by

$$i \sim_{(N,v)} j \text{ iff } i \neq j \text{ and } v(S \cup i) = v(S \cup j) \text{ for all } S \subseteq N \setminus \{i, j\}.$$

When $i \sim_{(N,v)} j$ holds, we say that i and j are symmetric in (N, v). Let $i \nsim_{(N,v)} j$ denote the negation of $i \sim_{(N,v)} j$. We omit the reference to (N, v) if it is clear from the context.

We introduce two standard axioms satisfied by ψ:

Efficiency, E For any $(N, v) \in \Gamma$, $\sum_{i \in N} \psi_i(N, v) = v(N)$.

Symmetry, S For any $(N, v) \in \Gamma$ and any $i, j \in N$ with $i \sim j$, $\psi_i(N, v) = \psi_j(N, v)$.

According to Thomson's (2012) categorization, symmetry is a punctual axiom because it is applied to each game separately. In this paper, we introduce a relational axiom that describes the fair treatment of symmetric players. To this end, we first focus on the following widely used fairness criterion:

[1] We denote a singleton set $\{i\}$ simply by i.

Balanced contributions property (Myerson (1980)) For any $(N, v) \in \Gamma$ and any $i, j \in N$,

$$\psi_i(N, v) - \psi_i(N \backslash j, v) = \psi_j(N, v) - \psi_j(N \backslash i, v).$$

In the above equation, the left-hand side represents j's contribution to i's payoff, while the right-hand side represents i's contribution to j's payoff. The balanced contributions property states that these contributions are balanced *between any two players*. In our new axiom, we relax this condition and require the balance of contributions only for symmetric players.

Balanced contributions property for symmetric players, BCS For any $(N, v) \in \Gamma$ and any $i, j \in N$ with $i \sim j$,

$$\psi_i(N, v) - \psi_i(N \backslash j, v) = \psi_j(N, v) - \psi_j(N \backslash i, v).$$

BCS states that if two players are in the same bargaining position in a game, then their contributions to each other's payoffs should be balanced.

3 Logical independence between BCS and S

The purpose of this section is to prove that BCS is independent of S even under E. This result means that BCS allows two symmetric players to receive different payoffs.

3.1 S does not imply BCS

We introduce a new solution $\tilde{\psi}$ defined by

$$\tilde{\psi}_i(N, v) = \begin{cases} \dfrac{v(N)}{n} & \text{if } j \sim k \text{ for some } j, k \in N, \\ v(N) & \text{if } j \nsim k \text{ for all } j, k \in N, \text{ and } i \leq j \text{ for all } j \in N, \\ 0 & \text{otherwise.} \end{cases}$$

One easily checks that $\tilde{\psi}$ satisfies E and S. To see that $\tilde{\psi}$ violates BCS, choose a player set $N \in \mathcal{N}$ with $\{1, 2, 3\} \subseteq N$ and consider the characteristic function $v : 2^N \to \mathbb{R}$ defined by

$$v(S) = \begin{cases} i & \text{if } S = \{i\} \text{ and } i \in N \backslash \{1, 3\}, \\ 1 & \text{otherwise.} \end{cases}$$

In this game, $1 \sim 3$ holds, but

$$\psi_1(N,v) - \psi_1(N\backslash 3, v) = \frac{1}{n} - 1 \neq \frac{1}{n} - 0 = \psi_3(N,v) - \psi_3(N\backslash 1, v).$$

3.2 BCS does not imply S

We introduce additional notations and lemmas. For each $(N,v) \in \Gamma$, we define $\mathcal{L}(N,v)$ and $\mathcal{S}(N,v)$ by

$$\mathcal{L}(N,v) = \{i \in N : v(i) \neq v(j) \text{ for all } j \neq i\},$$
$$\mathcal{S}(N,v) = \{i \in N : \text{ there exists } j \text{ such that } i \sim j\}.$$

Let $\to$ denote the binary relation on Γ defined by

$$(N,v) \to (N',w) \text{ iff there exists } i \in \mathcal{S}(N,v) \text{ such that } (N\backslash i, v) = (N',w).$$

Let $\to^*$ denote the transitive closure of $\to$. We define $f : \Gamma \to \mathbb{N}$ by

$$f(N,v) = \min\{|S| : (N,v) \to^* (S,v)\}.$$

In other words, $f(N,v)$ is the smallest number of players $|S|$ such that (S,v) can be obtained from (N,v) by eliminating symmetric players.

Let (N,v) be a game and let $\pi : N \to \mathbb{N}$ be an injection. We define $(\pi N, \pi v)$ by $\pi v(\pi S) = v(S)$ for all $S \subseteq N$, $S \neq \emptyset$. We say that (N',w) is equivalent to (N,v) iff there exists an injection $\pi : N \to \mathbb{N}$ such that $\pi(N) = N'$ and $\pi v = w$. It is clear that if (N',w) is equivalent to (N,v), then $f(N',w) = f(N,v)$.

Lemma 1. *Let $(N,v) \in \Gamma$. Then, for any $i,j \in \mathcal{S}(N,v)$, $f(N\backslash i, v) = f(N\backslash j, v)$.*

Proof. If $n = 2$, the result trivially holds. Suppose that $n = 3$. Let $N = \{i,j,k\}$ and $i,j \in \mathcal{S}(\{i,j,k\},v)$. Since $(\{i,k\},v)$ is equivalent to $(\{j,k\},v)$, we obtain $f(\{i,k\},v) = f(\{j,k\},v)$.

We proceed by an induction. Suppose that the result holds for (N',v), $|N'| = t-1$, and we prove the result for (N,v), $|N| = t$, where $t \geq 4$.

Let $i,j \in \mathcal{S}(N,v)$. If $i \sim_{(N,v)} j$, then $(N\backslash i, v)$ is equivalent to $(N\backslash j, v)$, which implies the desired equation. Suppose that $i \nsim_{(N,v)} j$. Then, there exist $i', j' \in N$ such that $i' \sim i$ and $j' \sim j$.

Since $j' \sim_{(N\backslash i,v)} j$, $\mathcal{S}(N\backslash i, v) \neq \emptyset$. From the definition of f, there exists $k \in \mathcal{S}(N\backslash i, v)$ such that

$$f(N\backslash i, v) = f(N\backslash\{i,k\}, v). \tag{1}$$

According to the induction hypothesis,

$$f(N\backslash\{i,j\},v) = f(N\backslash\{i,k\},v). \tag{2}$$

Similarly, since $i' \sim_{(N\backslash j,v)} i$, $\mathcal{S}(N\backslash j,v) \neq \emptyset$. From the definition of f, there exists $k' \in \mathcal{S}(N\backslash j,v)$ such that

$$f(N\backslash j,v) = f(N\backslash\{j,k'\},v). \tag{3}$$

According to the induction hypothesis,

$$f(N\backslash\{i,j\},v) = f(N\backslash\{j,k'\},v). \tag{4}$$

Equations (1) to (4) imply the desired equation. □

Lemma 2. *Let $(N,v) \in \Gamma$. Then, for any $i,j \in \mathcal{S}(N,v)$,*

$$f(N,v) = f(N\backslash i,v) = f(N\backslash j,v).$$

Proof. Let $i,j \in \mathcal{S}(N,v)$. From the definition of f, there exists $k \in \mathcal{S}(N,v)$ such that

$$f(N,v) = f(N\backslash k,v).$$

From Lemma 1,

$$f(N\backslash i,v) = f(N\backslash k,v) = f(N\backslash j,v).$$

The above equations imply the desired equations. □

Lemma 3. *Let $(N,v) \in \Gamma$ with $n \geq 3$. If $f(N,v) = 2$, then $|\mathcal{L}(N,v)| = 0$ or 1.*

Proof. Suppose, on the contrary, that there exists a game $(N,v) \in \Gamma$ with $n \geq 3$ such that $f(N,v) = 2$ and $|\mathcal{L}(N,v)| \geq 2$. Choose $i,j \in \mathcal{L}(N,v)$, $i \neq j$. Since $f(N,v) = 2$, there exists $(\{i',j'\},v) \in \Gamma$ such that $(N,v) \to^* (\{i',j'\},v)$. Suppose that $i \notin \{i',j'\}$. Then, there exists $S \subseteq N$ with $|S| \geq 2$ such that

$$(N,v) \to^* (S,v) \to^* (\{i',j'\},v),\ i \sim_{(S,v)} k \text{ for some } k \in S.$$

This contradicts the fact that $v(i) \neq v(k)$. Thus, we must have $i \in \{i',j'\}$. Following the same argument, we obtain $j \in \{i',j'\}$. In other words, $(\{i',j'\},v) = (\{i,j\},v)$. Since $n \geq 3$, there exists $k' \in N$ such that $i \sim_{(\{i,j,k'\},v)} k'$ or $j \sim_{(\{i,j,k'\},v)} k'$. Then, we must have $v(i) = v(k')$ or $v(j) = v(k')$. In either case, we obtain a contradiction with $i,j \in \mathcal{L}(N,v)$. □

We are ready to define a solution that satisfies E and BCS, but violates S. Consider

the following solution $\hat{\psi}$:
<u>$n = 1$</u>**:** $\hat{\psi}(\{i\}, v) = v(i)$.
<u>$n = 2$</u>**:** Let $N = \{i, j\}$. Let $\delta = |v(i) - v(j)|$ and we define

$$\begin{aligned} \hat{\psi}_i(\{i,j\}, v) &= \frac{(i-j)\cdot\delta + v(\{i,j\})}{2}, \\ \hat{\psi}_j(\{i,j\}, v) &= \frac{(j-i)\cdot\delta + v(\{i,j\})}{2}. \end{aligned} \tag{5}$$

<u>$n \geq 3$</u>**:** Let $(N, v) \in \Gamma$. If $f(N, v) \neq 2$ or $|\mathcal{L}(N, v)| \neq 1$, we define

$$\hat{\psi}_i(N, v) = \frac{v(N)}{n} \text{ for all } i \in N. \tag{6}$$

Suppose that $f(N, v) = 2$ and $|\mathcal{L}(N, v)| = 1$. Let $\{k\} = \mathcal{L}(N, v)$ and we define

$$\hat{\psi}_i(N, v) = \begin{cases} \dfrac{i \cdot |v(i) - v(k)|}{2} & \text{if } i \neq k, \\ v(N) - \displaystyle\sum_{j \in N \setminus \{k\}} \dfrac{j \cdot |v(j) - v(k)|}{2} & \text{if } i = k. \end{cases} \tag{7}$$

To see that $\hat{\psi}$ violates S, choose a player set $N \in \mathcal{N}$ with $\{1, 2, 3\} \subseteq N$ and consider the characteristic function $v : 2^N \to \mathbb{R}$ defined by

$$v(S) = |S \setminus \{3\}| \text{ for all } S \subseteq N.$$

In this game, $1 \sim 2$ holds, but $\hat{\psi}_1(N, v) = 1/2 \neq 1 = \hat{\psi}_2(N, v)$.

Proposition 1. *$\hat{\psi}$ satisfies BCS.*

Proof. Let $(N, v) \in \Gamma$ and $i, j \in N$ with $i \sim j$.
Case 1: Suppose that $n = 2$. Since $v(i) = v(j)$, from (5),

$$\hat{\psi}_i(\{i,j\}, v) - \hat{\psi}_j(\{i,j\}, v) = \hat{\psi}_i(\{i\}, v) - \hat{\psi}_j(\{j\}, v) = 0.$$

Case 2: Suppose that $n = 3$. Let $N = \{i, j, k\}$. We consider two subcases.[2]
Subcase 2-1: Suppose $f(N, v) = 1$. In this case, we must have $v(k) = v(i) = v(j)$, which implies $|\mathcal{L}(N, v)| = 0$. From (5) and (6),

$$\hat{\psi}_i(\{i,j,k\}, v) - \hat{\psi}_j(\{i,j,k\}, v) = \hat{\psi}_i(\{i,k\}, v) - \hat{\psi}_j(\{j,k\}, v) = 0.$$

Subcase 2-2: Suppose $f(N, v) = 2$. Since $f(N, v) = 2$, we must have $v(k) \neq v(i)$, which

[2]We remark that if $n = 3$ and $i \sim j$, then $f(N, v) \neq 3$.

implies $|\mathcal{L}(N,v)| = 1$. From (5) and (7)

$$\begin{aligned}
&\hat{\psi}_i(\{i,j,k\},v) - \hat{\psi}_j(\{i,j,k\},v) = \frac{(i-j)\cdot|v(i)-v(k)|}{2},\\
&\hat{\psi}_i(\{i,k\},v) - \hat{\psi}_j(\{j,k\},v)\\
&= \frac{(i-k)\cdot|v(i)-v(k)| + v(\{i,k\})}{2} - \frac{(j-k)\cdot|v(j)-v(k)| + v(\{j,k\})}{2}\\
&= \frac{(i-j)\cdot|v(i)-v(k)|}{2}.
\end{aligned}$$

Case 3: Suppose that $n \geq 4$ and $f(N,v) \neq 2$. From Lemma 2, $f(N\backslash i,v) \neq 2$ and $f(N\backslash j,v) \neq 2$. Thus, from (6),

$$\hat{\psi}_i(N,v) - \hat{\psi}_j(N,v) = \hat{\psi}_i(N\backslash j,v) - \hat{\psi}_j(N\backslash i,v) = 0.$$

Case 4: Suppose that $n \geq 4$ and $f(N,v) = 2$. From Lemma 2, $f(N\backslash i,v) = f(N\backslash j,v) = 2$. From Lemma 3, $|\mathcal{L}(N,v)| = 0$ or 1. We consider three subcases.
Subcase 4-1: Suppose that $|\mathcal{L}(N,v)| = 1$. Let $\{k\} = \mathcal{L}(N,v)$. From the definition of $\mathcal{L}(N,v)$, $k \in \mathcal{L}(N\backslash i,v)$ and $k \in \mathcal{L}(N\backslash j,v)$. Since $f(N\backslash i,v) = f(N\backslash j,v) = 2$, together with Lemma 3, we obtain $\{k\} = \mathcal{L}(N\backslash i,v) = \mathcal{L}(N\backslash j,v)$. Thus, from (7),

$$\hat{\psi}_i(N,v) - \hat{\psi}_j(N,v) = \hat{\psi}_i(N\backslash j,v) - \hat{\psi}_j(N\backslash i,v) = \frac{(i-j)\cdot|v(i)-v(k)|}{2}.$$

Subcase 4-2: Suppose that $|\mathcal{L}(N,v)| = 0$ and $|\mathcal{L}(N\backslash i,v)| = 0$. Since $(N\backslash i,v)$ is equivalent to $(N\backslash j,v)$, we must have $|\mathcal{L}(N\backslash j,v)| = 0$. From (6),

$$\hat{\psi}_i(N,v) - \hat{\psi}_j(N,v) = \hat{\psi}_i(N\backslash j,v) - \hat{\psi}_j(N\backslash i,v) = 0.$$

Subcase 4-3: Suppose that $|\mathcal{L}(N,v)| = 0$ and $|\mathcal{L}(N\backslash i,v)| = 1$. We first prove that $\{j\} = \mathcal{L}(N\backslash i,v)$; suppose, on the contrary, that $\{k\} = \mathcal{L}(N\backslash i,v)$ for some $k \neq j$. Since $|\mathcal{L}(N,v)| = 0$, we must have $v(i) = v(k)$. Together with $v(i) = v(j)$, we obtain $v(j) = v(k)$. This contradicts $\{k\} = \mathcal{L}(N\backslash i,v)$.

Since $(N\backslash i, v)$ is equivalent to $(N\backslash j, v)$, we have $\{i\} = \mathcal{L}(N\backslash j, v)$. From (6) and (7),

$$\begin{aligned}
\hat{\psi}_i(N,v) - \hat{\psi}_j(N,v) &= \frac{v(N)}{n} - \frac{v(N)}{n} = 0, \\
\hat{\psi}_i(N\backslash j, v) - \hat{\psi}_j(N\backslash i, v) &= v(N\backslash j) - \sum_{\ell \in N\backslash\{i,j\}} \frac{\ell \cdot |v(\ell) - v(i)|}{2} \\
&\quad - \Big\{ v(N\backslash i) - \sum_{\ell \in N\backslash\{i,j\}} \frac{\ell \cdot |v(\ell) - v(j)|}{2} \Big\} \\
&= 0,
\end{aligned}$$

which completes the proof. □

Remark 1. The following stronger cousin of S is often discussed in the literature:

Anonymity, A For any $(N, v) \in \Gamma$ and any injection $\pi : N \to \mathcal{U}$, $\psi(\pi(N), \pi(v)) = \pi(\psi(N, v))$.

One can verify that A is stronger than BCS. To see this, let ψ satisfy A and let $(N, v) \in \Gamma$ and $i, j \in N$ with $i \sim j$. From A, $\psi_i(N, v) = \psi_j(N, v)$. Moreover, since $(N\backslash j, v)$ is equivalent to $(N\backslash i, v)$, again from A, $\psi_i(N\backslash i, v) = \psi_j(N\backslash i, v)$. These two equations establish that $\psi_i(N, v) - \psi_i(N\backslash j, v) = \psi_j(N, v) - \psi_j(N\backslash i, v)$. ∎

4 Application of BCS to axiomatizations

In this section, we apply BCS to axiomatizations and find that in axiomatizations of an anonymous solution, replacing S with BCS results in new axiomatizations of the solution.

Theorem 1. *Let φ be a solution satisfying A. Suppose that φ is characterized by a set of axioms including E and S. Then, φ is also characterized by the set of axioms in which S is replaced with BCS.*

Proof. Since φ satisfies A, φ satisfies BCS (see Remark 1). Conversely, let ψ be a solution satisfying the set of axioms in which S is replaced with BCS. We proceed by an induction on the number of players.

Induction basis: From E, $\psi(N, v) = \varphi(N, v)$ holds for 1-person games.

Induction step: Let $k \geq 1$. Suppose that $\psi(N, v) = \varphi(N, v)$ holds for k-person games, and we prove the result for $k + 1$-person games.

It suffices to prove that ψ satisfies S for the class of $k+1$-person games.[3] Let $(N, v) \in \Gamma$

[3]We say that ψ satisfies S for the class of $k + 1$-person games if for any $(N, v) \in \Gamma$ with $n = k + 1$ and $i, j \in N$ with $i \sim j$, $\psi_i(N, v) = \psi_j(N, v)$.

with $n = k + 1$ and $i, j \in N$ with $i \sim_{(N,v)} j$. From BCS,

$$\psi_i(N, v) - \psi_j(N, v) = \psi_i(N\backslash j, v) - \psi_j(N\backslash i, v). \tag{8}$$

According to the induction hypothesis,

$$\psi_i(N\backslash j, v) - \psi_j(N\backslash i, v) = \varphi_i(N\backslash j, v) - \varphi_i(N\backslash i, v). \tag{9}$$

Since $i \sim_{(N,v)} j$, $(N\backslash j, v)$ is equivalent to $(N\backslash i, v)$. From the A of φ,

$$\varphi_i(N\backslash j, v) = \varphi_j(N\backslash i, v). \tag{10}$$

(8), (9) and (10) imply $\psi_i(N, v) = \psi_j(N, v)$. Since the choices of (N, v) and i, j are arbitrary, we conclude that ψ satisfies S. An induction with respect to k concludes that $\psi(N, v) = \varphi(N, v)$ for all $(N, v) \in \Gamma$. □

Remark 2. In the statement of Theorem 1, we do not require that A is included in the set of axioms characterizing the solution φ. ∎

As shown in Section 3, BCS and S are essentially different axioms even under E. However, Theorem 1 states that when combined with other axioms, BCS and S have the same implication. Moreover, most solutions in the literature satisfy E and A, and hence Theorem 1 can be applied to them. From these results, we conclude that existing solutions are fair solutions in terms of both punctual and relational fairness for symmetric players.

We conclude this paper by applying Theorem 1 to previous axiomatizations, restricting our attention to the Shapley value and the equal division value.

Corollary 1. *A solution ψ satisfies E, BCS, the null player property*[4]*, and additivity (see Shapley (1953)) if and only if $\psi = Sh$.*

Corollary 2. *A solution ψ satisfies E, BCS, and marginality (see Young (1985)) if and only if $\psi = Sh$.*

Corollary 3. *A solution ψ satisfies E, BCS, and coalitional strategic equivalence (see Chun (1989, 1991)) if and only if $\psi = Sh$.*

Corollary 4. *A solution ψ satisfies E, BCS, additivity, and the nullifying player property (see van den Brink (2007)) if and only if $\psi = ED$.*

Corollary 5. *A solution ψ satisfies E, BCS, and coalitional standard equivalence (see van den Brink (2007)) if and only if $\psi = ED$.*

[4]The null player property states the following: Let $(N, v) \in \Gamma$ and $i \in N$. If $v(S \cup i) - v(S) = 0$ for all $S \subseteq N\backslash i$, then $\psi_i(N, v) = 0$.

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