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Abstract

It is well known that the production efficiency theorem fails under the optimal nonlinear income tax model with endogenous wages. Thus, production efficiency is thought to hold when wages are exogenous. However, we show that the desirability of production efficiency depends on the credibility of the government's commitment. If the government cannot commit to its tax policies and the types of taxpayers are separated completely, then the production efficiency theorem should be violated in an optimal solution, and this result holds even if wages are exogenous.

Keywords Income taxation, Production efficiency, Commitment
JEL classification D82, H21, H54

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1 Introduction

A celebrated result of Diamond and Mirrlees (1971) clarifies the relationship between a distortionary tax and economic efficiency on the production side. They show that the marginal rates of technical substitution across sectors should be equal under linear commodity taxation. However, several recent studies identify cases in which production efficiency does not hold when the government uses nonlinear income taxation to mitigate income inequalities, in line with Mirrlees (1971) analysis of the second-best economy. According to Naito (1999), Pirttilä and Tuomala (2001), Blackorby and Brett (2004), and Gaube (2005a,b), an engine of this production inefficiency is the endogeneity of wages, as in Stiglitz (1982). Therefore, the fixed wage assumption is crucial to obtain production efficiency under nonlinear income taxation.

Our study presents a novel factor in production inefficiency, time consistency, which appears when we extend the static model of nonlinear income taxation to a multi-period setting. Using information revealed by taxpayers in a given period, the government has an incentive to re-optimize its redistribution policy in a future period, so the government's commitment to its tax policy is not credible, and assuming the government can commit to this policy is problematic. Knowing that the government cannot credibly commit, taxpayers may adjust their behavior to conceal information, which is called the ratchet effect.

A growing body of research considers the issue of time consistency in the case of optimal nonlinear income taxation (see Apps and Rees (2006), Bisin and Rampini (2006), Brett and Weymark (2008), Krause (2009), Guo and Krause (2011, 2013), and Berliant and Ledyard (2014)). For example, Brett and Weymark (2008) and Guo and Krause (2011) show that when the ratchet effect is taken into account, the Atkinson and Stiglitz (1976) theorem, which states that optimal nonlinear income taxation makes commodity taxation useless when leisure and consumption are separable, does not hold. Morita (2016) investigates the optimal provision rule of public goods under nonlinear income taxation in the absence of commitment and provides implications for the provision rule, including the ratchet effect. However, none of these studies analyzes the impact of the ratchet effect on public spending on intermediate inputs. To the best of our knowledge, this study is the first to examine production efficiency in the case where the government cannot credibly commit.

The basic framework of our model follows Pirttilä and Tuomala (2001), but it differs in three main ways from their setting. First, we suppose that the relative wage rate is constant over time periods to avoid an endogenous wage effect due to income taxation. This assumption distinguishes this study from Naito (1999). Second, we focus only on the case in which public investment augments private sector production but does not provide direct consumption benefits.[1] Third, for simplicity, we construct a simple

[1] As in this setting, Pirttilä and Tuomala (2005) and Gaube (2005b) assume that public spending does not affect individuals' utilities by considering public spending as an intermediate input. In

two-period model and do not use the overlapping generations framework.

We find that the production efficiency theorem should be modified in cases when the government cannot commit to its tax policy and the skill types of individuals are completely separated. The marginal effect of investment in public and private capital on the incentive compatibility constraint is a key factor in this result. We show that if public investment is more (less) complementary with labor inputs than private investment is, then public investment tightens (relaxes) the incentive compatibility constraint more than investment in private capital does. Thus, the marginal benefit of public capital should be lower (higher) than that of private capital.

The organization of this paper is as follows. Section 2 describes the framework of the basic model and considers the commitment case as the benchmark result. Section 3 examines the impact of the ratchet effect on production efficiency, and section 4 presents the numerical results. Section 5 offers concluding remarks.

2 Model

2.1 The individuals

We construct a model in which individuals live for two periods, $t = 1, 2$. There are two types of individuals, $i = 1, 2$, with n^1 low-income earners (type 1 individuals) and n^2 high-income earners (type 2 individuals) supplying labor in both periods l_t^i, consuming a private good in both periods c_t^i, and saving in the first period s^i. The utility function, which is assumed to be identical between individuals, is defined as follows:

$$U^i(c_1^i, c_2^i, l_1^i, l_2^i) = \sum_t \beta^{t-1}\{u(c_t^i) - v(l_t^i)\}, \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor. It is assumed that $u(\cdot)$ is a increasing concave function and that $v(\cdot)$ is a increasing convex function.

2.2 The producer

The production side of this economy utilizes capital K_t, two types of labor L_t^i, and a public input G_t. As is typical in the optimal taxation literature, the production sector consists of one competitive firm. The production function is defined as follows:

$$y_t = F(L_t^1, L_t^2, K_t, G_t) = f(L_t, K_t, G_t) \quad t = 1, 2, \tag{2}$$

where $L_t \equiv \theta^1 L_t^1 + \theta^2 L_t^2$ for $t = 1, 2$ and θ^i is a positive parameter indicating the productivities of the individuals, with $\theta^1 < \theta^2$. Equation (2) is assumed to be strictly

contrast, Sandmo and Dréze (1971), Edwards (1986), and Boadway and Keen (1993) consider public investment that affects consumption benefits.

monotone, twice continuously differentiable, strictly quasi-concave, and linearly homogeneous in labor and capital. Under the last assumption, the firm's profit is zero in equilibrium. Furthermore, suppose that the public input augments the output and does not rival the other inputs. Taken together, these assumptions imply

$$f_{G_t} \equiv \frac{\partial f}{\partial G_t}(\cdot) > 0, \; f_{L_t G_t} \equiv \frac{\partial^2 f}{\partial L_t \partial G_t}(\cdot) > 0, \text{ and } f_{L_t K_t} \equiv \frac{\partial^2 f}{\partial L_t \partial K_t}(\cdot) > 0.$$

2.3 Equilibrium

The labor market equilibrium conditions are

$$L_t^i = l_t^i n^i \quad i, t = 1, 2.$$

Let $K_1 = 0$. Then, the corresponding capital market equilibrium condition is

$$K_2 = s^1 n^1 + s^2 n^2 + s^G,$$

where s^G is government saving. Note that the savings technology available to the government is the same as that available to individuals. The commodity market equilibrium conditions are

$$y_t = c_t^1 n^1 + c_t^2 n_t^i \quad t = 1, 2.$$

In each period, the competitive firm maximizes its profit, $f(\cdot) - (1+r)K_t - \sum_i \omega_t^i L_t^i$, where $1+r$ and ω_t^i are the input prices in period t. The first-order conditions for labor inputs can be expressed as follows:

$$\frac{\partial f}{\partial L_t^i}(\cdot) = \frac{\partial f}{\partial L_t}(\cdot)\theta^i = \omega_t^i \quad i, t = 1, 2.$$

This result implies that the relative wage rate, $\omega_t \equiv \frac{\omega_t^1}{\omega_t^2}$, is constant across time periods. The first-order condition with respect to the capital input is

$$\frac{\partial f}{\partial K_2}(\cdot) = 1 + r.$$

2.4 The Government

The objective of the government is to maximize a utilitarian social welfare function,

$$\sum_t U^i(\cdot) n^i. \tag{3}$$

The government can observe the labor and capital incomes of both types of individuals, but it cannot observe their productivities. This informational assumption is similar to

that in the standard optimal income taxation problem, as in Mirrlees (1971). Thus, the government can set the nonlinear labor income tax $T_t(\omega^i l_t^i)$ and the nonlinear capital income tax $\tau(rs^i)$ to finance investment in public capital I_G and government savings s^G. The public capital is given by the following stock variables:

$$G_2 = I_G + (1 - \delta_G)G_1,$$

where $\delta^G \in (0, 1)$ is the depletion rate and G_1 is the constant. The government budget constraints are

$$\sum_i T_1(\omega^i l_1^i)n^i - I_G - s^G = 0 \tag{4}$$

$$\sum_i T_2(\omega^i l_2^i)n^i + \tau(rs^i)n^i + (1 + r)s^G = 0. \tag{5}$$

Since the production function exhibits constant return to scale, equations (4) and (5) can be rewritten as the following expressions:

$$y_1 - \sum_i c_1^i n^i - \sum_i s^i n^i - s^G = I_G \tag{6}$$

$$y_2 - \sum_i c_2^i n^i = 0. \tag{7}$$

These equations give the resource constraints in each period. The nonlinear income taxes are policies intended to achieve the allocation that satisfies the following incentive compatibility constraints:

$$U^1(c_1^1, c_2^1, l_1^1, l_2^1) \geq U^1(c_1^2, c_2^2, \frac{l_1^2}{\omega_1}, \frac{l_2^1}{\omega_2}) \tag{8}$$

$$U^2(c_1^2, c_2^2, l_1^2, l_2^2) \geq U^2(c_1^1, c_2^1, l_1^1\omega_1, l_2^1\omega_2). \tag{9}$$

We focus on the case where income is redistributed from type 2 individuals to type 1 individuals. Then, equation (9) is binding.

2.5 The commitment case

If the government can commit to a redistribution policy in the second period, then it cannot exploit information revealed in the first period to re-optimize its policy in the second period. The government chooses c_t^i, l_t^i, s^i (for all i and t), s^G, and I_G to

maximize equation (3) subject to equations (6), (7), and (9). The first-order conditions with respect to this planning problem give:

$$f_{G_2} = \frac{\lambda_1^C}{\beta\lambda_2^C} \quad \text{and} \quad f_{K_2} = \frac{\lambda_1^C}{\beta\lambda_2^C},$$

where λ_t^C denotes the Lagrange multiplier for the resource constraint in period t in the commitment case. These equations imply that the marginal productivities of private and public capital are equal through the ratio of the marginal social value of public funds across time, expressed by $\frac{\lambda_1^C}{\beta\lambda_2^C}$. Then, we obtain the following:

$$\frac{f_{G_2}}{f_{K_2}} = 1.$$

Several studies, such as Naito (1999), Pirttilä and Tuomala (2001), Gaube (2005a), show that in the presence of endogenous wages, deviating from production inefficiency can Pareto-improve social welfare. However, because the relative wage rate is constant in our model, production efficiency should hold.

3 The non-commitment case

When the relationship between the government and individuals continues for more than two periods, the government's commitment is not credible. Since the labor and capital incomes of individuals are observable, this information allows the government to infer the individuals' identities at the beginning of the second period. Thus, the government can re-optimize its redistribution policy in the second period to eliminate distortion. Knowing that the government can re-optimize its policy, however, the individuals can adjust their decision making in the first period, an effect known as the ratchet effect. This analysis focuses on the case where all type 2 individuals are separated in the first period, which is called the complete separation case.

3.1 The complete separation case

If individuals make different choices in the first period, then the government has sufficient information to carry out personalized lump-sum taxation in the second period and achieve full redistribution. Given $\mathbf{v} = (I_G, s_G, s^1, s^2)$, the government chooses c_2^i and l_2^i (for all i) to maximize

$$\sum_i \{u(c_2^i) - v(l_2^i)\} n^i$$

subject to equation (7). Let $V_2^S(\mathbf{v})$ be the value function in the second period. In the first period, the government chooses c_1^i, l_1^i, s^i (for $i = 1, 2$), I_G, and s^G to maximize

$$\sum_i \{u(c_1^i) - v(l_1^i)\}n^i + \beta V_2^S(\mathbf{v})$$

subject to equation (6) and the incentive compatibility constraint

$$U^2(c_1^2, c_2^2(\mathbf{v}), l_1^2, l_2^2(\mathbf{v})) \geq U^2(c_1^1, c_2^1(\mathbf{v}), \omega_1 l_1^1, \omega_2 l_2^1(\mathbf{v})), \tag{10}$$

where $c_2^2(\mathbf{v})$ and $l_2^2(\mathbf{v})$ are solutions of the second period problem for type 2 individuals and $c_2^1(\mathbf{v})$ and $l_2^1(\mathbf{v})$ are solutions of the second period problem for type 1 individuals. Here, let $f_{L_2G_2}\Delta$ denote the marginal effect of public capital on the incentive compatibility constraint (equation (10)) and $f_{L_2K_2}\Delta$ denote that of private capital. The short notation Δ is a common term (see (A.11) in the Appendix).

Combining these equations with the first-order conditions gives:

$$f_{G_2} = \frac{\lambda_1^S - \lambda_2^S f_{L_2G_2}\Delta}{\beta\lambda_2^S + f_{L_2}u''\Delta} \quad \text{and} \quad f_{K_2} = \frac{\lambda_1^S - \lambda_2^S f_{L_2K_2}\Delta}{\beta\lambda_2^S + f_{L_2}u''\Delta}, \tag{11}$$

where λ_t^S denotes the Lagrange multiplier for the resource constraint in period t under the non-commitment case. In contrast with the commitment case, the marginal productivities are different from each other, and neither is equal to the ratio of the marginal social value of public funds across time (i.e., $\frac{\lambda_1^S}{\beta\lambda_2^S}$). To understand this result, we can rewrite equation (11) as follows:

$$\frac{f_{G_2}}{f_{K_2}} = \frac{\lambda_1^S - \lambda_2^S f_{L_2G_2}\Delta}{\lambda_1^S - \lambda_2^S f_{L_2K_2}\Delta}. \tag{12}$$

In the Appendix, we show that Δ is negative. Intuitively, public and private capital both tighten the incentive compatibility constraint. If the cross derivative $f_{L_2G_2}$ is equal to $f_{L_2K_2}$, the marginal effect of public investment on the incentive compatibility constraint corresponds to that of private investment, and, therefore, it is optimal to choose public investment such that the marginal productivity of public investment equals that of private investment. On the other hand, if public capital is more complementary to the labor input than private capital is, the opportunity cost of public investment is larger than that of private investment, and vice versa. We obtain the following proposition.

Proposition 1. *A solution to the non-commitment nonlinear income taxation problem with complete separation in the first period satisfies the following:*

- *Production efficiency should hold, that is,* $\frac{f_{G_2}}{f_{K_2}} = 1$, *if public capital and private capital are complements of the labor input to the same degree.*

- *Production efficiency should not hold, that is,* $\frac{f_{G_2}}{f_{K_2}} \neq 1$*, if public capital is either more or less complementary to the labor input than private capital is.*

Proposition 1 implies that there are cases in which the production efficiency theorem does not prevail even if individuals' utility functions are additive and separable between consumption and leisure and the relative wage rate is constant. The key factors driving this result are the lack of commitment and the shape of the production function.

4 Numerical examples

In order to clarify the effect of the lack of commitment on the level of public and private capital, we present a numerical example. The production function (equation (2)) takes the constant elasticity of substitution form:

$$f(L_t, K_t, G_t) = G_t(a(K_t)^{-\sigma} + b(L_t)^{-\sigma})^{-\frac{1}{\sigma}},\ t = 1, 2,$$

where $a+b = 1$, $a > 0$, $b > 0$, and $\sigma \geq -1$. In line with Marrero (2008), we set $a = 0.42$ and $b = 0.58$. We assume that labor is a perfect substitute for private capital, that is, $\sigma = -1$. In this case, production inefficiency is desirable since the following holds:

$$0 = f_{L_t K_t} < f_{L_t G_t} = b,\ t = 1, 2.$$

Suppose that the initial level of public capital is unity (i.e., $G_1 = 1$). Both types of individuals have the same utility function (equation (1)), which is represented by:

$$U^i(\cdot) = \sum_t \beta^{t-1}\left(\frac{1}{1-\eta}(c_t^i)^{1-\eta} - \frac{1}{1+\gamma}(l_t^i)^{1+\gamma}\right).$$

We assume that the parameters η and γ are unity, as in Bisin and Rampini (2006). Thus, the subutility function is a log utility. For simplicity, we assume that no individuals have the option to store the private good in the first period. Since most individuals work for 40 years of their lives, we take each period to be 20 years in length. Assuming an annual discount rate of 5%, the 20-year discount factor is $\beta = 0.38$, which is consistent with Guo and Krause (2013). Following Marrero (2008), the marginal costs of both private and public investment are unity, and the annual depreciation factor is 0.038, so that the 20-year depreciation factor is $\delta_G = 0.521$. According to OECD Statistics, the unit labor cost in OECD countries was two in 2016, so we normalize type 1 individuals' parameter θ_1 to unity, and type 2 individuals' parameter is assumed to be $\theta_2 = 2$. The simulation results are illustrated in Table 1.

The highest level of social welfare can be achieved in the commitment case because the government faces more constraints in the non-commitment case than in the commitment case. Furthermore, Table 1 shows that the level of public capital in the

	Social welfare	I_G	s_G
The commitment case	-0.663978	2.47489	-4.0614
The non-commitment case	-1.13575	0.920373	-3.89555

Table 1: Simulation results

non-commitment case is lower than that in the commitment case. Under this production function, equation (12) can be rewritten as:

$$f_{G_2} = f_{K_2}(1 - \frac{\lambda_2^S}{\lambda_1^S} f_{L_2 G_2} \Delta).$$

This result follows because the labor and capital inputs are perfectly substitutable and there is no marginal effect of private capital on the incentive compatibility constraint, that is, $f_{L_2 K_2} \Delta = 0$. The left-hand side is the marginal benefit of public capital, and the right-hand side is the opportunity cost of public capital. An investment in public capital, however, tightens the incentive compatibility constraint, which raises the opportunity cost of public capital and increases the investment in private capital. Thus, in the non-commitment case, the level of public investment is lower but government saving is higher relative to the commitment case.

5 Conclusion

Using the framework of a two-period optimal nonlinear income tax model, we find that the desirability of production efficiency crucially relies on the government's ability to credibly commit. If the government can commit to a redistribution policy in the first period, production efficiency is optimal. In other words, the marginal productivities of public and private investment must be equal. However, if the government cannot credibly commit to a policy, production efficiency will be violated to relax the incentive constraint unless the complementarity between public investment and labor supply coincides with the complementarity between private investment and labor supply.

This study challenges the robustness of the production efficiency theorem obtained under an exogenous wage by investigating the issue of time consistency in the case of a nonlinear income tax problem with two periods. Naito (1999) demonstrates that production efficiency fails under an endogenous wage and holds under an exogenous wage. In contrast, this analysis argues that even if the wage is fixed, production efficiency is not necessary when the government cannot credibly commit. Thus, the important contribution of this study is the finding that the assumption of a fixed wage is no longer important to maintain production efficiency when the government's commitment is not credible.

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Appendix

Let $\mathcal{L}_2^S$ be the Lagrangian for the optimization problem in the second period. The first-order conditions are:

$$\frac{\partial \mathcal{L}_2^S}{\partial c_2^i} = u'(c_2^i) - \lambda_2^S = 0 \quad i = 1, 2 \tag{A.1}$$

$$\frac{\partial \mathcal{L}_2^S}{\partial l_2^i} = -v'(l_2^i) + \lambda_2^S f_L \theta^i = 0 \quad i = 1, 2 \tag{A.2}$$

$$\frac{\partial \mathcal{L}_2^S}{\partial \lambda_2^S} = f(n^1\theta^1 l_2^1 + n^2\theta^2 l_2^2, K_2, G_2) - \sum_i n^i c_2^i = 0. \tag{A.3}$$

Equation (A.1) yields $c_2^1 = c_2^2$, and equation (A.2) yields $l_2^1 < l_2^2$. Let $\mathcal{L}_1^S$ be the Lagrangian function in the first period. The first-order conditions with respect to I_G and s_G are

$$\frac{\partial \mathcal{L}_1^S}{\partial I_G} = \beta \frac{\partial V_2^S}{\partial I_G} - \lambda_1^S + \beta \phi_1^2 [-v'(l_2^2(\mathbf{v})) \frac{dl_2^2}{dI_G} + v'(\omega_2 l_2^1(\mathbf{v}))\omega_2 \frac{dl_2^1}{dI_G}] = 0 \tag{A.4}$$

$$\frac{\partial \mathcal{L}_1^S}{\partial s^G} = \beta \frac{\partial V_2^S}{\partial s^G} - \lambda_1^S + \beta \phi_1^2 [-v'(l_2^2(\mathbf{v})) \frac{dl_2^2}{ds^G} + v'(\omega_2 l_2^1(\mathbf{v}))\omega_2 \frac{dl_2^1}{ds^G}] = 0. \tag{A.5}$$

By the envelope theorem, we have

$$\frac{\partial V_2^S}{\partial I_G} = \lambda_2^S f_{G_2}, \quad \frac{\partial V_2^S}{\partial s^G} = \lambda_2^S f_{K_2}. \tag{A.6}$$

The bordered Hessian associated with equations (A.1) to (A.3) is

$$A \equiv \begin{pmatrix} u''(c_2^1) & 0 & 0 & 0 & -1 \\ 0 & u''(c_2^2) & 0 & 0 & -1 \\ 0 & 0 & -v''(l_2^1) + \lambda_2^S f_{L_2L_2} n^1 (\theta^1)^2 & \lambda_2^S f_{L_2L_2} n^2 \theta^1 \theta^2 & f_{L_2}\theta^1 \\ 0 & 0 & \lambda_2^S f_{L_2L_2} n^1 \theta^1 \theta^2 & -v''(l_2^2) + \lambda_2^S f_{L_2L_2} n^2 (\theta^2)^2 & f_{L_2}\theta^2 \\ -n^1 & -n^2 & n^1 f_{L_2}\theta^1 & n^2 f_{L_2}\theta^2 & 0. \end{pmatrix}$$

Its determinant is

$$|A| = (f_{L_2})^2 (u''(c_2^2))^2 \sum_{i \neq j} v''(l_2^i) n^j (\theta^j)^2 - u''(c_2^2)\{v''(l_2^1)v''(l_2^2) - \lambda_2^S f_{L_2L_2} \sum_{i \neq j} v''(l_2^i) n^j (\theta^j)^2\}.$$

By the concavity of $u(\cdot)$ and $f(\cdot)$ and the convexity of $v(\cdot)$, $|A|$ is positive. By Cramer's rule, we obtain:

$$\frac{dl_2^i}{dI_G} = \frac{-1}{|A|} u''(c_2^i) \frac{v'(l_2^i)v''(l_2^j)}{\lambda_2^S f_{L_2}} \{\lambda_2^S f_{L_2G_2} + f_{G_2} f_{L_2} u''(c_2^i)\} \quad i,j = 1,2, i \neq j \tag{A.7}$$

$$\frac{dl_2^i}{ds^G} = \frac{-1}{|A|} u''(c_2^i) \frac{v'(l_2^i)v''(l_2^j)}{\lambda_2^S f_{L_2}} \{\lambda_2^S f_{L_2K_2} + f_{K_2} f_{L_2} u''(c_2^i)\} \quad i,j = 1,2, i \neq j. \tag{A.8}$$

Substituting equations (A.6) to (A.8) into equations (A.4) and (A.5), respectively, yields:

$$f_{G_2}[\beta \lambda_2^S + f_{L_2} u''(c_2^2)\Delta] = \lambda_1^S - \lambda_2^S f_{L_2G_2}\Delta \tag{A.9}$$

$$f_{K_2}[\beta \lambda_2^S + f_{L_2} u''(c_2^2)\Delta] = \lambda_1^S - \lambda_2^S f_{L_2K_2}\Delta, \tag{A.10}$$

where

$$\Delta \equiv \frac{\beta \phi_1^2 u''(c_2^1)}{|A| \lambda_2^S f_{L_2}} \left(v'(l_2^2)^2 v''(l_2^1) - v'(\omega l_2^1)\omega v'(l_2^1) v''(l_2^2) \right). \tag{A.11}$$

Suppose that the dis-utility is an iso-elastic $v(l_t^i) = \frac{\kappa}{\gamma} l^\gamma$ for $\kappa > 0$ and $\gamma > 1$. In this case, Δ can be rewritten as

$$\Delta \equiv \frac{\beta \phi_1^2 u''(c_2^1)}{|A| \lambda_2^S f_{L_2}} \kappa^3 (\gamma - 1)(l_2^1 l_2^2)^{\gamma - 2} [(l_2^2)^\gamma - (\omega l_2^1)^\gamma].$$

Equation (A.2) yields $l_2^1 < l_2^2$. Then, Δ is negative. Dividing equation (A.9) by equation (A.10) yields equation (12). □