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Principle of Least Action in Contemporary Physics and Economics

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Abstract

The principle of least action means that every object in the universe take the shortest course in its motion. For example, light and point mass will move on the straight line without refraction or external force.

This is, however, a very strange thing because moving objects in the nature often have no intentions. Even so, mathematical physics has been built taking a basis on this principle including relativity theories and quantum mechanics.

Contemporary macroeconomics also shares mathematical background with physics. Economics originally shed light on rational behavior of human beings. Human beings have their minds and calculate their actions. Because of this, mathematical formalization of economics is suitable for the principle of least action.

This paper summarizes logical correspondences between contemporary physics and economics.

Frank Plumpton Ramsey published a paper on optimal growth theory. John Maynard Keynes, who was the editor of the journal at the time, proposed use of principle of variations to Ramsey. In fact, this was the first paper which paraphrased principle of least action explicitly in economics.

Several years later, Keynes published his monumental book. It is probably true that Keynes was very conscious of Einstein's works when he wrote the sentences. After all, Keynes's book treated an economy under the real effect from money while Einstein thought about the motion of point mass under the relativistic effects.

Stochastic disturbance of the economy came to play a substantial role in economic growth around 1980. The effects from shocks sustain long and move the growth path into a new one. This image is very akin to Feynman's path integral, where a particle takes every possible path at the same time.

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Introduction

The principle of least action means that every object in the universe take the shortest course in its motion. For example, light and point mass will move on the straight line without refraction or external force.

This is, however, a very strange thing because moving objects in the nature often have no intentions. Even so, mathematical physics has been built taking a basis on this principle including relativity theories and quantum mechanics.

Contemporary macroeconomics also shares mathematical background with physics. Economics originally shed light on rational behavior of human beings. Human beings have their minds and calculate their actions. Because of this, mathematical formalization of economics is suitable for the principle of least action.

This paper summarizes logical correspondences between contemporary physics and economics.

1. Origin of Principle of Least Action

Pierre-Louis Moreau de Maupertuis proposed principle of least action in his two papers[1]. In the first paper, he introduced the principle in the area of optics. Two years later, he published another paper where the principle was extended into all phenomena in the nature.

He intended to prove the existence of God who control the movements of particles in the world. He thought that there was God's behavior of optimization behind the principle of least action.

Only a few months later of Mauperyuis's first paper, Leonhard Euler published a book[2]. He calculated a parabola orbit of the object in appendix 2 of the book. The orbit is optimized following the equation $\int ds\sqrt{v}$. He substituted to this the relations $v=a+gx$ and $ds=dx\sqrt{(1+pp)}$ which were reduced from the equation of motion $dv=gdx$ into this. He got y=(2/g)√{C(a-G+gx)} applying Euler equation of calculus of variations.

In the summer of 1755, Joseph-Louis Lagrange of age 19 sent a letter[3] to Euler. It suggested an algebraic expression of principle of least action.

He introduced variation symbol δ and expressed the general principle of motion like this;

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1 Maupertuis (1744, 1766).

2 Euler (1744).

3 Lagrange (1760-61).

$$\delta \int u ds = 0 \quad (1).$$

We can separate the left-hand side of the equation (1) into two parts.

$$\int \delta u ds + \int u \delta ds = 0 \quad (2)$$

The first term can be formalize using the relation *u=ds/dt* as follows;

$$\int \delta u ds = (X\delta x + Y\delta y + Z\delta z)\frac{dt}{M} \quad (3).$$

The second term can also be formalize using the relation $ds=\sqrt{(dx^2+dy^2+dz^2)}$ as follows;

$$\int u\delta ds = \int u\left(\frac{dx}{ds}d\delta x + \frac{dy}{ds}d\delta y + \frac{dz}{ds}d\delta z\right) \quad (4).$$

$$-\int\left(d\frac{udx}{ds}\delta x + d\frac{udy}{ds}\delta y + d\frac{udz}{ds}\delta z\right)$$

When the equation (1) is held for any variations of x, y and z, we can obtain the three equations (5). These are the equations of motion of the particle *M*.

$$d\frac{udx}{ds} - \frac{X}{M}dt = 0, d\frac{udy}{ds} - \frac{Y}{M}dt = 0, d\frac{udz}{ds} - \frac{Z}{M}dt = 0 \quad (5)$$

Lagrange made the principle of least action the only method for deducing any equation of motion. He accomplished that following the principle of variations.

2. Lagrange and Hamilton Formats

When we apply Lagrangian method to Newtonian mechanics, we suppose a physical amount Lagrangian.

$$L = L(q, \dot{q}) = T(q, \dot{q}) - U(q) \quad (6)$$

Here T, U and q are kinetic energy, potential energy and generalized coordinate respectively.

From D'Alembert's principle, we obtain the equation (7).

$$\sum_{k=1}^{K}\left(\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}^k} - \frac{\partial L(q,\dot{q})}{\partial q^k}\right)\delta q^k = 0 \quad (7)$$

Because the contents of the bracket must be zero independently, the equation (8) follows.

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}^k} - \frac{\partial L(q,\dot{q})}{\partial q^k} = 0 \quad (8)$$

This is Euler-Lagrange equation which we have already seen.

Instead of using the concepts of energy, we shall suppose the amount called Lagrangian, an abstract function of coordinates, time differential of coordinates and time itself. Then we can define the principle of least action like this. A particle moves along the path *q(t)* through which the action like the equation (9) possesses the minimum value.

$$S = \int_{t_A}^{t_B} L(q, \dot{q}, t)dt \qquad (9)$$

This generalization of Lagrangian made the principle of least action applicable for broader areas including economics.

In our world, time has uniformity. We can suppose the physical amount that is independent from time.

$$\frac{\mathrm{dL}}{dt} = \frac{\partial L}{\partial t} + \sum_{k=1}^{K}\left(\frac{\partial L}{\partial q^k}\frac{dq^k}{dt} + \frac{\partial L}{\partial \dot{q}^k}\frac{d\dot{q}^k}{dt}\right) = \sum_{k=1}^{K}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k}\right)\dot{q}^k + \frac{\partial L}{\partial \dot{q}^k}\frac{d\dot{q}^k}{dt}\right) \qquad (10).$$

$$= \frac{d}{dt}\left(\sum_{k=1}^{K}\dot{q}^k\frac{\partial L}{\partial \dot{q}^k}\right) = \frac{d}{dt}\left(\sum_{k=1}^{K}p^k\dot{q}^k\right)$$

We define Hamiltonian, a physical amount looking at the equation (10). Here p is generalized momentum. The concept of Hamiltonian was proposed by William Rowan Hamilton in his paper in 1834[4].

$$H(q,p) \equiv \sum_{k=1}^{K}\dot{q}^k\frac{\partial L}{\partial \dot{q}^k} - L = \sum_{k=1}^{K}p^k\dot{q}^k - L \qquad (11)$$

As the relation *H=2T-L=2T-(T-U)=T+U* shows us, Hamiltonian equals to the total of kinetic energy and potential energy. In economics, Hamiltonian corresponds to the total utility of the nation, which is the sum of utility from consumption and the capital converted to utility.

Formally the total differential of Hamiltonian is like this;

$$dH = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial p}dq + \frac{\partial H}{\partial p}dt \qquad (12).$$

On the other hand, the same dH can be expressed using the definition as follows;

$$dH = \dot{q}dp + pd\dot{q} - dL = \dot{q}dp + pd\dot{q} - \left(\frac{\partial L}{\partial q}dq + \frac{\partial L}{\partial \dot{q}}d\dot{q} + \frac{\partial L}{\partial t}dt\right) \qquad (13).$$

$$= \dot{q}dp + pd\dot{q} - \dot{p}dq - pd\dot{q} - \frac{\partial L}{\partial t}dt = \dot{q}dp - \dot{p}dq - \frac{\partial L}{\partial t}dt$$

Comparing the equations (12) and (13), we can obtain canonical equations (14).

$$\frac{dq^k}{dt} = \frac{\partial H}{\partial p^k}, \frac{dp^k}{dt} = \frac{\partial H}{\partial q^k} \qquad (14).$$

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4 Hamilton (1834).

In addition to that, we can see the following relation.

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{15}$$

The equation (15) tells that Hamiltonian is integral of motions in the system.

Using the definition of Hamiltonian, we can represent the action like this;

$$S = \int_{t_A}^{t_B} [p\dot{q} - H(p,q)]dt \tag{16}$$

We shall take variations which are independent for *p* and *q*.

$$\begin{aligned} \delta S &= \int_{t_A}^{t_B} \left(\delta p\dot{q} + p\frac{d}{dt}\delta q - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q\right)dt \\ &= p\delta q|_{t_A}^{t_B} + \int_{t_A}^{t_B} \left[\left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q\right]dt \end{aligned} \tag{17}$$

Supposing $\delta S=0$, we obtain canonical equations again.

3. Principle of Least Action in Quantum Mechanics

Richard Phillips Feynman published a paper based on his doctoral thesis in 1948[5]. He invented path integral as a new approach to quantum mechanics. In quantum mechanics, a particle like an electron takes not only one path but also any other paths that are possible from *a* to *b*. Every path can be recognized only by the probabilities attached to each paths.

The probability can be expressed using a weight $\varphi[x(t)]$. The action $S[x(t)]$ corresponds to a path $x(t)$. We assign a phase $2\pi S/h$ to each path. Here h is Planck constant.

$$\varphi[x(t)] \propto exp\left(i\frac{S[x(t)]}{\hbar}\right) \tag{18}$$

We also introduced the concept of probability amplitude $K(b, a)$.

$$K(b,a) = \sum_{\forall x(t)} \varphi[x(t)] \tag{19}$$

The probability by which the particle moves from point a to point b is given as $K(b, a)$ squared.

$$P(b,a) = K(b,a)^2 \tag{20}$$

The principle of least action in classical physics is the limit case of path integral in quantum mechanics. This discovery gives us a philosophical interpretation of the principle of least action. In economics, economic agents intendedly optimize their behaviors. In this

[5] Feynman (1948).

meaning, it is natural that the principle of least action is held in an economic world. In comparison to that, it is very unnatural to suppose Nature optimizes its action. The formalization above instead shows us that Nature realizes every possible path just mechanically. As a result, the phases offset each other and only the classical trajectory remains.

Suppose the particle pass the point c on way from a to b, we can express probability amplitude like this;

$$K(b,a)=\int_{-\infty}^{\infty}dx_c K(b,c)K(c,a) \quad (21).$$

Dividing the time interval from a to b into N parts, we repeat the same operation and obtain this equation (22).

$$K(b,a)=\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_{N-1}\prod_{n=0}^{N-1}K(n+1,n) \quad (22)$$

We can approximate *K(n+1,n)* as follows;

$$K(n+1,n)=\frac{1}{A}\exp\left[\frac{i}{\hbar}S(n+1,n)\right] \quad (23).$$

$$\approx\frac{1}{A}\exp\left[i\frac{\Delta t}{\hbar}L(\frac{x_{n+1}-x_n}{\Delta t},\frac{x_{n+1}+x_n}{2},\frac{t_{n+1}+t_n}{2})\right]$$

After all, probability amplitude results in the equation (24).

$$K(b,a)=\lim_{N\to\infty}\frac{1}{A^N}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_{N-1}\,exp\left[\frac{i}{\hbar}\sum_{n=0}^{N-1}S(n+1,n)\right] \quad (24)$$

Here we represent $t'=t+\Delta t$ and $x'=x+\Delta x$. Then we substitute $\psi(x,t)$ into the equation (21) as $K(x,t;x_0,t_0)$.

$$\psi(x',t')=\int_{-\infty}^{\infty}dxK(x',t';x,t)\psi(x,t) \quad (25)$$

We substitute the equation (23) into the equation (25).

$$\psi(x',t+\Delta t)\approx\int_{-\infty}^{\infty}dx\frac{1}{A}exp\left[i\frac{\Delta t}{\hbar}L(\frac{x'-x}{\Delta t},\frac{x'+x}{2},t+\frac{\Delta t}{2})\right]\psi(x,t) \quad (26)$$

When Lagrangian is of one dimension, the equation (27) follows.

$$\Delta tL\left(\frac{x'-x}{\Delta t},\frac{x'+x}{2},t+\frac{\Delta t}{2}\right)=\frac{m(x'-x)^2}{2\Delta t}-\mathrm{U}\left(\frac{x'+x}{2},t+\frac{\Delta t}{2}\right)\Delta t \quad (27)$$

Here we replace $x'-x$ by ξ and represent x' as x once again. Then we expand the equation (26) and remain the first-order term for Δt.

$$\psi(x,t+\Delta t)=\int_{-\infty}^{\infty}d\xi\frac{1}{A}exp\left[i\frac{m\xi^2}{2\hbar\Delta t}-i\frac{\Delta t}{\hbar}U\left(x+\frac{\xi}{2},t+\frac{\Delta t}{2}\right)\right]\psi(x-\xi,t) \quad (28)$$

From the equation (28), we obtain the equation (29).

$$\psi(x,t)+\Delta t\frac{\partial\psi}{\partial t}=\int_{-\infty}^{\infty}d\xi\frac{1}{A}e^{\frac{im\xi^2}{2\hbar\Delta t}}\left[1-i\frac{\Delta t}{\hbar}U(x,t)\right]\left[\psi(x,t)-\xi\frac{\partial\psi}{\partial x}+\frac{\xi^2}{2}\frac{\partial^2\psi}{\partial x^2}\right] \quad (29)$$

When we make $\Delta t\rightarrow 0$ and $\xi\rightarrow 0$, both sides of the equation (29) must be $\psi(x, t)$. Because of this, we obtain the equation (30).

$$A=\int_{-\infty}^{\infty}d\xi e^{\frac{im\xi^2}{2\hbar\Delta t}}=\sqrt{\frac{2\pi i\hbar\Delta t}{m}} \quad (30)$$

Taking into account the equation (30) and the relations (31), we obtain the equation (32).

$$\int_{-\infty}^{\infty}d\xi\frac{1}{A}\xi e^{\frac{im\xi^2}{2\hbar\Delta t}}=0,\int_{-\infty}^{\infty}d\xi\frac{1}{A}\xi^2 e^{\frac{im\xi^2}{2\hbar\Delta t}}=\frac{i\hbar\Delta t}{m} \quad (31)$$

$$\psi+\Delta t\frac{\partial\psi}{\partial t}=\psi-i\frac{\Delta t}{\hbar}U(x,t)+\frac{i\hbar\Delta t}{2m}\frac{\partial^2\psi}{\partial x^2} \quad (32)$$

The equation (32) results in the equation (33).

$$i\hbar\frac{\partial\psi}{\partial t}=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+U(x,t)\right]\psi \quad (33)$$

The equation (33) is Schrödinger equation of wave function. This means that principle of least action forms the basis of quantum mechanics.

4. Principle of Least Action in Real Business Cycle Model

In economics, the method of analytical mechanics had been introduced very naturally. The principle of least action is also easily explained from the optimizing behavior of households and firms.

Real business cycle models proposed a new view of economic fluctuation. In the models, the economy chooses a new path stochastically.[6]

Now we suppose a very simple economy. The production function is Cobb-Douglas type.

$$Y_t=U_tK_t^a, 0<\mathrm{a}<1 \quad (34)$$

Here Y, U and K are national income, total factor productivity and capital stock respectively. Labor supply is given and normalized as unity. Stochastic shock happens in the productivity.

Suppose the capital stock is wholly depreciated in one term, the equation $K_{t+1}=I_t=S_t$ holds.

[6] Kydland=Prescott (1882).

Consumers live for only two terms. The present value of utility for whole life is like this;

$$\mathrm{u}(C_t, C_{t+1}) = \ln C_t + \frac{E[lnC_{t+1}]}{1+\theta} \quad (35).$$

Budget constraint of consumer is as follows;

$$\mathrm{C}_t + \frac{C_{t+q}}{1+r} = \mathrm{w}_t = (1-a)U_t K_t^a \quad (36).$$

When the consumer maximizes the present value of utility, the equation (37) follows.

$$\mathrm{S}_t = \mathrm{K}_{t+1} = \frac{w_t}{2+\theta} = \frac{(1-a)U_t K_t^a}{2+\theta} \quad (37)$$

The equation (37) results in the equation (38).

$$\mathrm{k}_{t+1} = b + ak_t + u_t \quad (38)$$

Here $k_t=ln(K_t)$, $u_t=ln(U_t)$ and $b=(1-a)/(2+\theta)$.

From the equation (34), the equation (39) follows.

$$\mathrm{y}_t = ak_t + u_t \quad (39)$$

From the equations (38) and (39), we obtain the equation (40).

$$\mathrm{y}_t = ab + y_{t-1} + u_t \quad (40)$$

When u_t is white noise, the relations (41) are held.

$$E(u_t) = 0, E(u_t^2) = \sigma^2, E(u_t u_{t-s}) = 0 \quad (41)$$

The economy follows the only path as in classical mechanics.

On the contrary to this, we suppose u_t is random walk with drift g. In this case, production shock u_t move the economy to a new path. This means that the economy has infinite number of possible paths as in quantum mechanics.

$$u_t = g + u_{t-1} + \varepsilon_t \quad (42)$$

Here ε_t is white noise.

5. Principle of Least Action in the General Theory of Relativity

In classical mechanics, space-time and objects do not interfere with each other. In the theory of relativity, however, objects or energy distort space-time and objects will move along distorted space-time.[7]

In the theory of relativity, an action can be divided into two parts. One represents distortion of space-time. The other comes from distribution of objects or energy.

$$\mathrm{S} = S_g + S_m = \int \mathcal{L}_g \sqrt{-g} d^4x + \int \mathcal{L}_m \sqrt{-g} d^4x \quad (43)$$

[7] Einstein (1916).

When $£_g = a + bR$ holds, the equation (44) follows. R is curvature scalar and g is the determinant of metric tensor $g_{\mu\nu}$.

$$\delta S_g = \delta \int \mathcal{L}_g \sqrt{-g} d^4x = a \int \delta(\sqrt{-g}) d^4x + b \int \delta(R\sqrt{-g}) d^4x \tag{44}$$

Here we use the relations (45) and (46).

$$\delta\left(a\sqrt{-g}\right) = -\frac{a}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \tag{45}$$

$$\begin{aligned}\delta\left(R\sqrt{-g}\right) &= (\delta R)\sqrt{-g} + R\delta\left(\sqrt{-g}\right) \\ &= \sqrt{-g}\delta\left(g^{\mu\nu}R_{\mu\nu}\right) - \frac{R}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} \\ &= \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g} g^{\mu\nu}\delta R_{\mu\nu} - \frac{R}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} \\ &= \left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\right)\sqrt{-g}\delta g^{\mu\nu} + \sqrt{-g} g^{\mu\nu}\delta R_{\mu\nu}\end{aligned} \tag{46}$$

Here $g^{\mu\nu}$ is inverse matrix of $g_{\mu\nu}$ and $R_{\mu\nu}$ is Ricci tensor.

After some calculations, we obtain the equation (47) from the equation (44).

$$\begin{aligned}\delta S_g &= a\int (a + bR)\sqrt{-g} d^4x \\ &= \int \left[-\frac{a}{2} g_{\mu\nu} + b\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\right)\right]\sqrt{-g}\delta g^{\mu\nu}\, d^4x \\ &= \int b\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R - \frac{a}{2b} g_{\mu\nu}\right)\sqrt{-g}\delta g^{\mu\nu}\, d^4x\end{aligned} \tag{44}$$

We call $-a/2b$ cosmological constant and represent as Λ.

Next we move to the calculation of δS_m.

$$\begin{aligned}\delta S_m &= \delta \int \mathcal{L}_m \sqrt{-g} d^4x = \int \left\{\frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial g^{\mu\nu}{}_{,\alpha}}\delta g^{\mu\nu}{}_{,\alpha}\right\} d^4x \\ &= \int \frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\, d^4x + \int \frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial g^{\mu\nu}{}_{,\alpha}}\partial_\alpha \delta g^{\mu\nu}{}_{,\alpha}\, d^4x \\ &= \int \frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\, d^4x + \int \partial_\alpha\left[\frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial g^{\mu\nu}{}_{,\alpha}}\right]\delta g^{\mu\nu}{}_{,\alpha}\, d^4x \\ &\quad - \int \partial_\alpha\left[\frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial g^{\mu\nu}{}_{,\alpha}}\delta g^{\mu\nu}{}_{,\alpha}\right] d^4x \\ &= \int \left\{\frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^\alpha}\frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial g^{\mu\nu}{}_{,\alpha}}\right\}\delta g^{\mu\nu} d^4x \\ &= \int -\frac{\sqrt{-g}}{2} T_{\mu\nu}\delta g^{\mu\nu}\, d^4x\end{aligned} \tag{45}$$

Here $T_{\mu\nu}$ is covariant tensor.

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left\{\frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^{\alpha}}\frac{\partial\left(\mathcal{L}\sqrt{-g}\right)}{\partial {g^{\mu\nu}}_{,\alpha}}\right\} \tag{46}$$

From these calculations, we obtain the equation (47).

$$\delta S = \delta S_g + \delta S_m = \int b\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R - \frac{a}{2b}g_{\mu\nu} - \frac{1}{2b}T_{\mu\nu}\right)\sqrt{-g}\,\delta g^{\mu\nu}\, d^4x \tag{47}$$

Because $\delta S=0$, we obtain the equation (48) defining $\Lambda\equiv -a/2b$ and $\kappa\equiv 1/2b$.

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{48}$$

The equation (48) is Einstein equation.

6. Principle of Least Action in Dynamic Stochastic General Equilibrium Model

In new Keynesian models, real economy and money interact with each other while money is neutral in neoclassical models. New Keynesian models consist from IS curve, Philips curve, Taylor rule, Fisher equation and rational expectations.[8]

$$Y_t = \bar{Y}_t - \alpha(r_t - \rho) + (Y_{t+1}^e - \bar{Y}_{t+1}) + \varepsilon_t \tag{49}$$

$$\pi_t = \beta\pi_{t+1}^e + \phi(Y_t - \bar{Y}_{t+1}) + \upsilon_t \tag{50}$$

$$i_t = \pi_t + \rho + \theta_\pi(\pi_t - \pi_t^*) + \theta_Y(Y_t - \bar{Y}_{t+1}) + \eta_t \tag{51}$$

$$r_t = i_t - \pi_{t+1}^e \tag{52}$$

$$\pi_{t+1}^e = E\pi_{t+1}, Y_{t+1}^e = EY_{t+1} \tag{53}$$

From the equations (49), (51), (52) and (53), we obtain dynamic aggregate demand curve.

$$Y_t = \bar{Y}_t + \frac{\alpha}{1+\alpha\theta_Y}E\pi_{t+1} - \frac{\alpha(1+\theta_\pi)}{1+\alpha\theta_Y}\pi_t + \frac{1}{1+\alpha\theta_Y}(EY_{t+1} - \bar{Y}_{t+1}) + \frac{\alpha\theta_\pi}{1+\alpha\theta_Y}\pi_t^* + \frac{1}{1+\alpha\theta_Y}\varepsilon_t + \frac{\alpha}{1+\alpha\theta_Y}\eta_t \tag{54}$$

We also obtain dynamic aggregate supply function from the equations (50) and (53).

$$\pi_t = \beta E\pi_{t+1} + \phi(Y_t - \bar{Y}_{t+1}) + \upsilon_t \tag{55}$$

From the equations (54) and (55), we obtain equilibrium values of national income and inflation rate.

[8] Blanchard=Kiyotaki (1987).

$$Y_t = \bar{Y}_t + (1-\beta\rho_\varepsilon)\psi_\varepsilon\varepsilon_t - \alpha(1+\theta_\pi-\rho_\upsilon)\psi_\upsilon\upsilon_t - \alpha\left(1+\beta\rho_\eta\right)\psi_\eta\eta_t + (1-\beta)\theta_\pi\psi_{\pi^*}\pi_t^* \tag{56}$$

$$\pi_t = \phi\psi_\varepsilon\varepsilon_t + (1+\alpha\theta_Y-\rho_\upsilon)\psi_\upsilon\upsilon_t - \alpha\phi\psi_\eta\eta_t + \phi\theta_\pi\psi_{\pi^*}\pi_t^* \tag{57}$$

$$\psi_\varepsilon = \frac{1}{[(1+\alpha\theta_Y-\rho_\varepsilon)(1-\beta\rho_\varepsilon)+\alpha\phi(1+\theta_\pi-\rho_\varepsilon)]} \tag{58}$$

$$\psi_\upsilon = \frac{1}{[(1+\alpha\theta_Y-\rho_\upsilon)(1-\beta\rho_\upsilon)+\alpha\phi(1+\theta_\pi-\rho_\upsilon)]} \tag{59}$$

$$\psi_\eta = \frac{1}{\left[\left(1+\alpha\theta_Y-\rho_\eta\right)\left(1-\beta\rho_\eta\right)+\alpha\phi\left(1+\theta_\pi-\rho_\eta\right)\right]} \tag{60}$$

$$\psi_{\pi^*} = \frac{1}{\phi\theta_\pi+(1-\beta)\theta_Y} \tag{61}$$

Here ρ_ε, ρ_υ and ρ_η are parameters expressing durability of demand shock, supply shock and monetary shock respectively.

Conclusion

Frank Plumpton Ramsey published a paper on optimal growth theory. John Maynard Keynes, who was the editor of the journal at the time, proposed use of principle of variations to Ramsey. In fact, this was the first paper which paraphrased principle of least action explicitly in economics.

Several years later, Keynes published his monumental book. It is probably true that Keynes was very conscious of Einstein's works when he wrote the sentences. After all, Keynes's book treated an economy under the real effect from money while Einstein thought about the motion of point mass under the relativistic effects.

Stochastic disturbance of the economy came to play a substantial role in economic growth around 1980. The effects from shocks sustain long and move the growth path into a new one. This image is very akin to Feynman's path integral, where a particle takes every possible path at the same time.

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