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Resolution of the n-person Prisoners' Dilemma by Kalai's Preplay Negotiation Procedure

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Abstract

In this paper, we apply the preplay negotiation procedure proposed by Kalai (1981) to the n-person prisoners' dilemma with $n \geq 3$ and examine whether it generates cooperation. We show that if the preplay is carried out at least once, the preplay game has a perfect equilibrium point that realizes cooperation. We also show that if the preplay is excuted at least twice, the game has a perfect equilibrium point that realizes cooperation regardless of the entering action profile (the starting point of the preplays). Our results are a generalization of Kalai's (1981) that were obtained for the two-person prisoners' dilemma.

JEL classification: C72; D62

Keywords: The n-person prisoners' dilemma; Preplay negotiation; Perfect equilibrium point

1 Introduction

People often fail to cooperate when they seek individual benefit. This is exemplified by environmental pollution, wasting of energy and/or resources, free-rider problems, *etc.* The n-person prisoners' dilemma is a strategic-form game that represents such conflict situations. In this game, players can choose either to cooperate or to defect, but they have a strong motivation to defect because doing so is a dominant action.

There are many studies exploring the possibility of cooperation in this game. They introduce a number of elements that are not captured in the game and examine whether

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cooperation is realized as a rational choice for players. Repetition of the game is one such element. There are numerous studies in this field, and they show that cooperation is attainable when the game is repeated infinitely or finitely many times under various conditions (Kreps *et al.* 1982, Neymann 1985, Fudenberg and Maskin 1986, Sekiguchi 1997, Ely and Valimaki 2002, Bhaskar *et al.* 2006, *etc.*). Okada (1993) introduced an option to include an enforcement agency for cooperation into the n-person prisoners' dilemma. Varian (1994) considered a compensation mechanism depending on the choices. Nishihara (1997, 1999 and 2008) introduced randomly ordered sequential choices and unobservability of cooperative actions. They showed that cooperation can be realized under certain conditions.

This paper focuses on Kalai's (1981) research. He proposed a preplay negotiation procedure that was carried out before the actual play of a game. The procedure consists of simultaneous moves by all players and a given number of preplays. At the beginning, all players choose actions simultaneously, which becomes the entering action profile of the first preplay. In the second and subsequent preplays, the outcome of the previous preplay becomes the entering action profile. Each preplay is carried out as follows. First, all players choose actions and if some players choose actions that differ from their entering actions, then their choices are fixed and the other players can choose actions again. For the second round of choices, if some players choose actions that differ from their entering actions, then their choices are fixed and the rest of the players can choose actions again, and so on. The preplay ends if all players' actions are fixed, or all players who can make choices choose the same actions as their entering actions. After the given number of preplays is finished, players execute their actions in the outcome of the final preplay, which is the actual play of the given game.

Kalai (1981) applied this procedure to the two-person prisoners' dilemma. He showed that if the preplay is carried out at least once, then the game has a perfect equilibrium point (hereafter PEP) proposed by Selten (1975), in which both players' actions are to cooperate in the final outcome of the preplay(s). Furthermore, he showed that if the preplay is executed twice or more, then the game has a PEP in which cooperation is realized in the final outcome regardless of which entering action profile is given first.

As Schelling (1978) pointed out, the n-person prisoners' dilemma captures important social-conflict situations, especially when $n \geq 3$. However, to date, it has not been clarified whether Kalai's (1981) preplay negotiation procedure works well for the n-person prisoners' dilemma with $n \geq 3$. The purpose of this paper was to examine this point.

To generalize Kalai's (1981) first result, we consider the following negotiation strategy. In the determination of the first entering action profile, the player chooses to cooperate, and in the preplay, she chooses to cooperate if and only if: (1) no player chose to defect before, or (2) the preplay is not the last and the choices of the other $n-1$ players are fixed to cooperate. We show, as Theorem 1, that the combination of this negotiation strategy is a PEP if the preplay is carried out at least once. In this equilibrium, all players' actions are to cooperate in the final outcome of the preplay(s).

To generalize Kalai's (1981) second result, we consider another negotiation strategy. In the determination of the first entering action, the player chooses the action that she arbitrarily selected in advance. In the first preplay, the player chooses to cooperate if and only if there is no player whose choice was fixed at the action to defect; in the later preplay, the player follows the negotiation strategy stated above. We show, as Theorem 2, that if the preplay is carried out at least twice, then the combination of this negotiation strategy is a PEP in the subgame that starts after arbitrarily entering action profile. In this equilibrium, all players' actions are to cooperate in the final outcome. We also show, as Theorem 3, that the combination of this negotiation strategy is a PEP. Thus, we generalize Kalai's (1981) results and verify that his procedure works well to resolve the n-person prisoners' dilemma for any $n \geq 2$.

This paper is organized as follows. In the next section, we formulate the n-person prisoners' dilemma and Kalai's (1981) preplay negotiation procedure. In Section 3, we show the difficulty of using the same analytical procedure as Kalai (1981) in the three or more-person case. In Section 4, we discuss the main results. In the final section, we conclude. All proofs are given in the appendix.

2 The Model

In this section, we formulate the n-person prisoners' dilemma and the preplay negotiation procedure proposed by Kalai (1981).

2.1 The n-person Prisoners' Dilemma

The n-person prisoners' dilemma is a strategic-form game $< N, \{C, D\}, (f_i)_{i \in N} >$, where $N = \{1, 2, ..., n\}$ $(n \geq 2)$ is the set of players; $\{C, D\}$ is each player's action set with the interpretation that C is *to cooperate* and D is *to defect*; $f_i : \{C, D\} \times \{0, 1, ..., n-1\} \to R$ is player i's payoff function. Here, $f_i(a, m)$ represents player i's von Neumann-Morgenstern utility when the player chooses $a \in \{C, D\}$ and m players other than i choose C. We assume the following A1 through A3 for all $i \in N$.

A1: $f_i(D, m) > f_i(C, m)$ for any $m \in \{0, 1, ..., n-1\}$.

A2: $f_i(C, n-1) > f_i(D, 0)$.

A3: $f_i(a, m)$ is increasing in m for each $a \in \{C, D\}$.

Assumption A1 states that player i will be better off if she chooses D rather than C regardless of the other players' choices. A2 means that if all players choose D, then the situation will be worse for player i compared with the situation in which they choose C. A3 implies that, regardless of her action, player i's payoff becomes greater as the number of players selecting C increases.

From A1, D strongly dominates C for all the players; however, if they choose D, then, by A2, such a situation is worse than the case in which all players choose C. Hence, the players face a "dilemma". This game is thought to represent various social-conflict situations.

We use the following terminology and notations. We call a combination of actions of all the players *an action profile*. For any $I \subseteq N$, $A(I)$ denotes the set of all the action combinations of the players I. For any $I \subseteq N$, any $a \in A(I)$ and any $i \in I$, a_i denotes player

i's action in a. For any $I \subseteq N$, any $a \in A(I)$ and any $J \subseteq I$, a_J denotes the combination of the actions of the players J in a. For any two disjoint sets of players I and I', and for any $a \in A(I)$ and $a' \in A(I')$, $a + a'$ denotes the concatenation of a and a', i.e., $(a + a')_I = a$ and $(a + a')_{I'} = a'$. For any $I \subseteq N$, $C(I)$ (resp. $D(I)$) denotes an action combination such that $a_i = C$ (resp. $a_i = D$) for all $i \in I$. Therefore, $C(I) + D(N - I)$ is an action profile in which players in I choose C and the other players choose D. When there is no possibility of confusion, we use the notation $C(I)$ as an abbreviation for $C(I) + D(N - I)$. For example, we say that an action profile is $C(I)$ to indicate that it is $C(I) + D(N - I)$. Finally, for any action profile a, any $I \subseteq N$ and any $b \in \{C, D\}$, we define $I_{ab} = \{i \in I : a_i = b\}$. That is, I_{ab} is the set of the players in I who choose b in a.

2.2 Kalai's Preplay Negotiation Procedure

Kalai (1981) proposed the following preplay negotiation procedure. Before the negotiation begins a positive integer k, which represents the number of preplays, is given and publicly announced. At the beginning of the procedure, all players choose actions simultaneously. After this move, players carry out a preplay k times. In each preplay, an action profile, which we call the *entering action profile*, is given and announced at the beginning. We describe the procedure of a preplay later. When a preplay is finished, an action profile, which we call *the outcome* of the preplay, is determined. In the first preplay, the entering action profile is the result of the first simultaneous moves. In the later preplays, the entering action profile is the outcome of the previous preplay. When k preplays are finished, the players must execute the actions of the final outcome as the play of the given strategic-form game, which is the n-person prisoners' dilemma in this paper.

In this paper, we formulate the preplay proposed by Kalai (1981) by *states* and their *transition rule*. A state is a pair (a, I) in which $a \in A(N)$ and $I \subseteq N$. The first state is (e, N), where e is the entering action profile. At each state (a, I), the players in I simultaneously choose actions. Their choices $a' \in A(I)$ are announced to all the players.

The state transition rule is the following.

R1: If $a_I = a'$ or if $a_i \neq a'_i$ for all $i \in I$, then the state transition ends and the outcome of this preplay is $a_{N-I} + a'$.

R2: Otherwise, the state proceeds to the next one $(a_{N-I} + a', I')$, where $I' = \{i \in I : a_i = a'_i\}$.

Regarding this procedure, Kalai (1981) makes two points. First, this is formalized as a modified version of the real procedure of housing trades in the USA. Second, this procedure has the following two advantages: (1) it is simple; and (2) the players obtain the least payoff in the original strategic-form game, which induces them to participate in this procedure.

Let Γ^k be the extensive form game which represents this situation. Information sets are defined as follows. For any $l \leq k$, we call $s = (a^0_{v_0}; a^1_1, a^1_2, ..., a^1_{v_1}; a^2_1, a^2_2, ..., a^2_{v_2}; ...; a^m_1, a^m_2, ..., a^m_{v_m})$ *an action sequence* if $a^p_q \in A(I)$ for any $p \in \{0, 1, 2, ..., m\}$, any $q \in \{1, 2, 3, ..., v_p\}$ and some $I \subseteq N$. We say s is *possible* if there is a sequence of states $t = ((b^1_1, I^1_1), (b^1_2, I^1_2), ...,(b^1_{v_1}, I^1_{v_1}); (b^2_1, I^2_1), ..., (b^2_{v_2}, I^2_{v_2}); ....; (b^m_1, I^m_1), ..., (b^m_{v_m}, I^m_{v_m}))$ which satisfies the following conditions: (1) $a^0_{v_0} = b^1_1 \in A(N)$; (2) $a^p_q \in A(I^p_q)$ for any p and q; (3) for any $p \leq m-1$, $(b^p_{v_p}, I^p_{v_p})$ and $a^p_{v_p}$ satisfy the ending rule R1 as a state and action choices, b^{p+1}_1 is the outcome stated in R1, and $I^{p+1}_1 = N$; (4) for any p and any $q \leq v_p - 1$, (b^p_q, I^p_q) and a^p_q do not satisfy R1 as a state and action choices, and (b^p_{q+1}, I^p_{q+1}) is the next state stated in R2; (5) if $m = k$, then $(b^m_{v_m}, I^m_{v_m})$ and $a^m_{v_m}$ do not satisfy R1.

These conditions mean that s and t can occur as a sequence of choices and a sequence of states, and that the preplay negotiation procedure continues after s. Precisely, the condition (1) means that $a^0_{v_0}$ is possible choices in the first simultaneous moves and that it is the entering action profile of the first preplay. The condition (2) means that a^p_q is possible choices in the moves of (a^p_q, I^p_q). The condition (3) implies that, if $p \leq m-1$, the pth preplay ends at the state $(b^p_{v_p}, I^p_{v_p})$, and the $(p+1)$th preplay begins with the state (b^{p+1}_1, I^{p+1}_1). The condition (4) means that if $q \leq v_q - 1$, pth preplay does not ends at (b^p_q, I^p_q) and (b^p_{q+1}, I^p_{q+1}) is the next state. The condition (5) implies that, if $m = k$, the state transition does not end at $(b^m_{v_m}, I^m_{v_m})$. Let $(a(s), I(s))$ be the state next to $(b^m_{v_m}, I^m_{v_m})$

when $a^m_{v_m}$ is chosen. Namely, if $(b^m_{v_m}, I^m_{v_m})$ and $a^m_{v_m}$ satisfy R1, then $a(s)$ is the outcome stated in R1, and $I(s) = N$; otherwise, $(a(s), I(s))$ is the next state indicated by R2.

Every player knows what actions have been chosen when she makes a choice during her move of a state. Hence, for each player $i \in I(s)$, the set of decision nodes in the move of $(a(s), I(s))$ is her information set when an action sequence s is realized. Let $P_i(s)$ be this information set. For each $i \in N$, define S_i as the set of all the possible action sequence s satisfying $i \in I(s)$. Let P_i^0 be the set of player i's decision nodes at which she chooses an action in the first simultaneous moves. The collection of player i's information sets is defined as $\mathscr{P}_i = \{P_i^0\} \cup \{P_i(s) : s \in S_i\}$. For any $l \leq k$, let S_i^l be the set of possible action sequence s such that $(a(s), I(s))$ is in the lth preplay and $i \in I(s)$. Define $\mathscr{P}_i^l = \{P_i(s) : s \in S_i^l\}$. This is the collection of player i's information sets which she reaches in the lth preplay.

For any $l \leq k$ and any $P \in \mathscr{P}_i^l$, there is a unique $s \in S_i^l$ such that $P = P_i(s)$, and for this s, there is a unique state $(a(s), I(s))$ in the lth preplay. Let $(a(P), I(P))$ denote this state. Namely, player i's choice at P is the choice during the move of $(a(P), I(P))$ in the lth preplay. We assume that the entire structure of Γ^k is common knowledge among the players.

We call a mapping from $\mathscr{P}_i$ to $\{C, D\}$ player i's *negotiation strategy*. We call a mapping from $\mathscr{P}_i - \{P_i^0\}$ to $\{C, D\}$ player i's *preplay strategy*. For any $e \in A(N)$, let $\Gamma^k(e)$ be the subgame of Γ^k that begins from the moves of the first preplay with the entering action profile e.

Let $\epsilon_1, \epsilon_2, \epsilon_3, ...$ be a sequence of positive real numbers that converge to zero. We define a perturbation of Γ^k using ϵ_z $(z = 1, 2, 3, ...)$ later. When a perturbation is given, we define the perturbed game of Γ^k.

When we evaluate players' expected payoffs in a perturbed game, we use *Landau symbols* $O(\epsilon_z^x)$ and $o(\epsilon_z^x)$. As an additional symbol, we define $O^+(\epsilon_z^x)$ as some function $g(\epsilon_z)$, which satisfies $\lim_{z\to\infty} \frac{g(\epsilon_z)}{\epsilon_z^x} = h$ for some $h > 0$.

3 Backward Elimination of Weakly Dominated Actions

Kalai (1981) investigated Γ^1 and $\Gamma^k(e)$ $(k \geq 2)$ for the two-person case. For Γ^1, he showed that there is a PEP that realizes (C, C) as the final outcome. For $\Gamma^k(e)$ $(k \geq 2)$, he showed that for any $e \in A(\{1, 2\})$, there is a PEP that realizes (C, C) as the final outcome.

He analyzed these games by backwardly eliminating weakly dominated actions. This is done for Γ^1 as follows. For any state $(a, \{i\})$, D is better than (hence, weakly dominates) C, hence C is eliminated. For any state $(a, \{1, 2\})$ for players 1 and 2, one action weakly dominates the other, hence the dominated action is eliminated. In the first simultaneous moves, C weakly dominates D for players 1 and 2, hence C is left to both players to choose. We call this procedure *backward elimination of weakly dominated actions* (abbreviated *BEWDA*).[1]

BEWDA is an excellent analytical procedure to find a PEP. However, in the case of three or more people, we cannot proceed in some states. To illustrate this, let us investigate Γ^1 using BEWDA for the following three-person case: $f_i(C, m) = 2m$; $f_i(D, m) = 2m + 1$ for $i = 1, 2, 3$ and $m = 0, 1, 2$.

For any state $(a, \{i\})$, D weakly dominates C for player i, hence C is eliminated. Consider the state $(a, \{1, 2\})$. When $a = (C, C, a_3)$, the final outcome is (C, C) if players 1 and 2 choose C; otherwise, it is (D, D). Therefore, C weakly dominates D for both players and D is eliminated. The same holds for any state (a, I) in which $I = \{i, j\}$ and $a_i = a_j = C$.

When $a = (C, D, a_3)$, the correspondence of the choices of players 1 and 2 to the final outcome (hereafter, the choice-outcome correspondence) is shown in Figure 1. In this figure, D weakly dominates C for Player 1, and D strongly (hence, weakly) dominates C for Player 2. Hence, C is eliminated for both players. When they choose D, the final outcome is (D, D, a_3). The same holds for any state (a, I) in which $I = \{i, j\}$ and $a_i \neq a_j$.

When $a = (D, D, a_3)$, the choice-outcome correspondence of players 1 and 2 is shown in Figure 2. This indicates that D strongly dominates C for both players. Hence, C is eliminated for both. When they choose D, the final outcome is (D, D, a_3). The same holds

[1] We use this terminology following Huang *et al.* (2014).

		Player 2	
		C	D
Player	C	(D, C, a_3)	(C, D, a_3)
	D	(D, C, a_3)	(D, D, a_3)

Figure 1: The choice-outcome correspondence in $((C, D, a_3), \{1, 2\})$.

for any state in which $I = \{i, j\}$ and $a_i = a_j = D$.

		Player 2	
		C	D
Player	C	(C, C, a_3)	(C, D, a_3)
	D	(D, C, a_3)	(D, D, a_3)

Figure 2: The choice-outcome correspondence in $((D, D, a_3), \{1, 2\})$.

Consider state $((C, C, C), \{1, 2, 3\})$. From the above analysis, the choice-outcome correspondence of players 1, 2 and 3 is shown in Figure 3. In this figure, there is no weak dominance for any player. For Player 1, D is better than C if players 2 and 3 choose C, but C is better than D if players 2 and 3 choose C and D, respectively. The same holds for the other players.

		Player 3: C				Player 3: D	
		Player 2				Player 2	
		C	D			C	D
Player 1	C	(C, C, C)	(C, D, C)	Player 1	C	(C, C, D)	(D, D, D)
	D	(D, C, C)	(D, D, D)		D	(D, D, D)	(D, D, D)

Figure 3: The choice-outcome correspondence in $((C, C, C), \{1, 2, 3\})$.

This example illustrates that BEWDA does not necessarily work well for the analysis of Γ_1 in the three or more-person case. Moreover, it seems that in such a case, Γ^1 has no PEP, whereby cooperation is realized. Such a conclusion, however, is premature. Although BEWDA is a superior procedure, it does not derive all PEPs. We can see this in the analysis of the following perturbed game.

Consider the following perturbation: for any state (a, I), if $I_{aC} = \{i\}$, then we put probability ϵ_z on the branch C of player i's move. On any other branch, we use a probability of ϵ_z^4. Under this perturbation, in $((C,C,D),\{1,2\})$, players 1 and 2 have a payoff matrix, as shown in Figure 4.

		Player 2: C	Player 2: D
Player 1	C	$4+o(\epsilon_z^3), 4+o(\epsilon_z^3)$	$1-O^+(\epsilon_z), 1+o(\epsilon_z^3)$
	D	$1+o(\epsilon_z^3), 1-O^+(\epsilon_z)$	$1+o(\epsilon_z^3), 1+o(\epsilon_z^3)$

Figure 4: The payoff matrix in $((C,C,D),\{1,2\})$ under perturbation.

When players 1 and 2 choose C and D, respectively, Player 1 gets the expected payoff $1-O^+(\epsilon_z)$ for the following reason. In this case, the next state is $((C,D,D),\{1\})$ with the probability that converges to 1 when $z \to \infty$. Player 1 will choose D in this state, because by doing so she can obtain a greater expected payoff than by choosing C. Hence, the final outcome is (D,D,D) with a probability that converges to 1 when $z \to \infty$. However, during the move of $((C,D,D),\{1\})$, C is realized with probability ϵ_z and the outcome is (C,D,D). Any other outcome has a probability of at most ϵ_z^4. Hence, Player 1's expected payoff is $f_1(D,0)+\epsilon_z\{f_1(C,0)-f_1(D,0)\}+o(\epsilon_z^3) = 1-O^+(\epsilon_z)$. Similarly, Player 2's expected payoff is $1-O^+(\epsilon_z)$ if Player 1 and 2 choose D and C, respectively.

Figure 4 shows two Nash equilibria: (C,C) and (D,D), if z is sufficiently large. We have a similar argument for each state $(a,\{i,j\})$, such that $a_i = a_j = C$ and $s_{N-\{i,j\}} = D$. Suppose that the equilibrium (D,D) is selected in these states. In state $((C,C,C),\{1,2,3\})$, players' choice-outcome correspondences are shown in Figure 5. This figure shows the final outcomes that occur with a probability that converges to 1 when $z \to \infty$.

Player 3: C

		Player 2: C	Player 2: D
Player 1	C	(C,C,C)	(D,D,D)
	D	(D,D,D)	(D,D,D)

Player 3: D

		Player 2: C	Player 2: D
Player 1	C	(D,D,D)	(D,D,D)
	D	(D,D,D)	(D,D,D)

Figure 5: The choice-outcome correspondence in $((C,C,C),\{1,2,3\})$ under perturbation.

Figure 5 indicates that the choice combination (C,C,C) is a Nash equilibrium if z is sufficiently large. Investigating Nash equilibria for other states, we show that there is a

PEP that realizes cooperation. In the next section, we develop this analysis.

4 Main Results

We consider the following preplay strategy.

Definition 1. *For each $i \in N$, define $\tau_i : \mathscr{P}_i - \{P_i^0\} \to \{C, D\}$ as follows: when $P \in \mathscr{P}_i^k$ or $|I(P)| \geq 2$, $\tau_i(P) = C$ iff $(a(P), I(P)) = (C(N), N)$; when $P \in \mathscr{P}_i^l$ for some $l \leq k-1$ and $I(P) = \{i\}$, $\tau_i(P) = C$ iff $a(P)_{N-\{i\}} = C(N - \{i\})$*

In this definition, the condition $(a(P), I(P)) = (C(N), N)$ means that D was not chosen before P, including the first simultaneous moves. Note that $I \neq N$ holds for $(C(N), I)$ only if the players in $N - I$ changed their actions from their entering actions D to C. Hence, roughly speaking, this preplay strategy assigns D when D is chosen before. The only exception is the case in which $P \in \mathscr{P}_i^l$ $(l \leq k-1)$, $I(P) = \{i\}$ and $a(P)_{N-\{i\}} = C(N-\{i\})$, i.e., player i can realize $C(N)$ only by her choice before the final preplay. Ignoring this case as an exception, we can say that τ_i is a "trigger"-like strategy.

We define perturbation η_z^1 for $z = 1, 2, 3, ...$ using ϵ_z. In the first simultaneous moves, we put ϵ_z^{4k} on each branch. For any state (a, I) of the lth preplay and any $i \in I$, we put perturbation probabilities on the branches of player i's move at (a, I) as follows: (1) If $I \neq N$ and $I_{aC} = \{i\}$, then we put ϵ_z on C; (2) If $l \geq 2$, $I = N$, $a_i = C$, and $|I_{aC}| \geq 2$, then we put ϵ_z^2 on C; (3) If $l \geq 2$, $I = N$, and $I_{aC} = \{i\}$, then we put ϵ_z^3 on C; (4) We put ϵ_z^{4k} on any other branch. For any $e \in A(N)$ and any $z = 1, 2, 3...$, let $\Gamma_z^k(e)$ be the perturbed game defined from $\Gamma^k(e)$ and η_z^1. Using $\Gamma_1^k(e), \Gamma_2^k(e), \Gamma_3^k(e), ...$ as a test sequence, we obtain the following proposition.

Proposition 1. *For any $k \geq 1$ and any $e \in A(N)$, $(\tau_1, ..., \tau_n)$ is a PEP of $\Gamma^k(e)$.*

When the players use $(\tau_1, ..., \tau_n)$, the final outcome is $C(N)$ if $e = C(N)$, otherwise it is $C(\phi)$. We define $C\tau_i$ as player i's negotiation strategy such that she chooses C during the first simultaneous moves and makes action choices by τ_i later. For $z = 1, 2, 3...$, let Γ_z^k

be the perturbed game defined from Γ^k and η_z^1. Using $\Gamma_1^k, \Gamma_2^k, \Gamma_3^k, ...$ as a test sequence, we have the next theorem from Proposition 1. The proof is straightforward.

Theorem 1. *For any $k \geq 1$, $(C\tau_1, ..., C\tau_n)$ is a PEP of Γ^k.*

If $C\tau_1, ..., C\tau_n$ are used in Γ^k, the final outcome is $C(N)$. Hence, this theorem shows that cooperation is achieved by a PEP if preplay is done at least once. This is an n-person version of Kalai's (1981) result for the single-preplay game.

The main feature of the perturbation η_z^1 is that players have *inertia* for choosing C. This inertia exists in the move of the first state of the second and subsequent preplays. It also exists in the move of the second and later states (a, I) in every preplay if the player is the only person whose action is C in a_I. The inertia decreases the expected payoff of each player $i \in I_{aC}$ if she chooses C, not following τ_i, when $|I_{aC}| \geq 2$. If there is no perturbation, C and D are equivalent. The same outcome occurs; hence, there is the same payoff, because even if the player chooses C, she can change that selection in the next move. In the perturbed game Γ_z^k, however, C is realized with a probability of ϵ_z during the next move against the player's will if she chooses C alone during the previous move. This decreases her expected payoff based on Assumption A1 if the preplay is the final one. If the preplay is not final, the choice C will still have a relatively large probability in later preplays, until the final preplay. Thus, choosing D is better than choosing C. We do not have to create any inertia of D for each player $i \in I_{aD}$ because there is no case that τ_i assigns C.

The inertia of C in the first state is given by the smaller probability ϵ_z^2 when there are two or more players whose entering action is C. This is the case for the following reason. Let the probability be ϵ_z^x. Consider Player 1's choice to be the first move of the final preplay when the entering action profile is $C(\{1, 3\})$. Now the value of τ_1 is D. Regardless of her choice, the final outcome is $C(\phi)$, with a probability that converges to 1 when $z \to \infty$. If Player 1 chooses C, then the final outcome is $C(\{1\})$ with a probability of $O^+(\epsilon_z)$ because of the inertia of C in their next move. This factor decreases Player 1's expected payoff. At the same time, however, the final outcome is $C(\{1, 3\})$ with a probability of $O^+(\epsilon_z^x)$, because of the inertia of C in Player 3's move. This factor increases Player 1's expected payoff if $f_1(C,1) > f_1(D,0)$. Thus, to ensure that D is better than C for Player 1, we have to make $x > 1$. The reason for the probability of ϵ_z^3 is more technical (see Footnote 2 in the appendix).

Next, we examine the PEP for $k \geq 2$ in the following preplay strategy.

Definition 2. *For any $i \in N$, we define $\sigma_i : \mathscr{P}_i - \{P_i^0\} \to \{C, D\}$ as follows: when $P \in \mathscr{P}_i^1$, $\sigma_i(P) = C$ iff $a(P)_{N-I(P)} = C(N - I(P))$; when $P \in \mathscr{P}_i^l$ for some $l \geq 2$, $\sigma_i(P) = \tau_i(P)$.*

The interpretation of σ_i in the first preplay is that player i chooses to cooperate if all players' cooperation is possible. The condition $a(P)_{N-I(P)} = C(N - I(P))$ means that if all players in $I(P)$ choose C, then cooperation is realized.

For any $e \in A(N)$, we define perturbation η_z^e for $z = 1, 2, 3, ...$ using ϵ_z, as follows. In the first simultaneous moves, we put ϵ_z^{4k} on each branch. In the first preplay, for any state (a, I) and any player $i \in I$, we put perturbation probabilities on player i's move, as follows: (1) if $I = N$ and $a_i \neq e_i$, then we assign ϵ_z on D; (2) if $I \neq N$ and i is the only player whose action is a_i in a_I, we put ϵ_z on branch a_i and ϵ_z^{4k} on any other branch. In later preplays, the perturbation probabilities are the same as η_z^1.

For $z = 1, 2, 3...$, let Γ_z^{ke} be the perturbed game defined from Γ^k and η_z^e. For any $e' \in A(N)$, let $\Gamma_z^{ke}(e')$ be the perturbed game, which is defined based on $\Gamma^k(e')$ and η_z^e. For the next theorem, we fix $\hat{e} \in A(N)$ arbitrarily and use $\Gamma_1^{k\hat{e}}(e), \Gamma_2^{k\hat{e}}(e), \Gamma_3^{k\hat{e}}(e), ...$ as a test sequence for a given e.

Theorem 2. *For any $k \geq 2$ and any $e \in A(N)$, $(\sigma_1, ..., \sigma_n)$ is a PEP of $\Gamma^k(e)$.*

This theorem shows that cooperation is realized by a PEP whatever entering action profile is given first. It implies that even if the first entering action profile is determined exogeneously, cooperation can be achieved. Kalai (1981) demonstrated this property for the two-person prisoners' dilemma. Theorem 2 is a generalization of Kalai's result to $n \geq 3$.

In η_z^e, players have *inertia* not only on C but also on D in the first preplay. The reason is similar to that explaining the inertia on C in η_z^1. Suppose that σ_i assigns C for player $i \in I_{aD}$. When $|I_{aD}| \geq 2$, even if the player chooses D, she has a chance to change it for C in the next move. To make choosing C preferable to choosing D, we add inertia on D in the player's move in the next state.

Finally, we study the PEP of Γ^k. Define $a\sigma_i : \mathscr{P}_i \to \{C, D\}$ as player i's negotiation strategy, which assigns $a \in \{C, D\}$ at the first simultaneous moves and the same choices as σ_i in the subsequent move. We can show the next theorem using $\Gamma_1^{ke}, \Gamma_2^{ke}, \Gamma_3^{ke}, ...$ as a test sequence.

Theorem 3. *For any $k \geq 2$ and any $e \in A(N)$, $(e_1\sigma_1, ..., e_n\sigma_n)$ is a PEP of Γ^k.*

From Theorem 2, it appears that whatever choice is assigned in the first simultaneous moves, if players use $\sigma_1, ..., \sigma_n$ in the subsequent move; the combination of such a negotiation strategy will be a PEP of Γ^k. Theorem 3 shows that this is true. In this theorem, we assume that player i arbitrarily selects e_i when determining the negotiation strategy. η_z^e $(z = 1, 2, 3, ...)$ is then constructed from $e = (e_1, ..., e_n)$.

Although e_i is selected arbitrarily by player i, that choice must be better than choosing $b \neq e_i$ at the first simultaneous moves for $(e_1\sigma_1, ..., e_n\sigma_n)$ to be a PEP. For this purpose, we make the perturbation probability of D large in player i's move in the first state (a, N) in which $a_i \neq e_i$. If the player chooses $a_i \neq e_i$ at the first simultaneous moves, then D will be realized with a probability of ϵ_z during the player's move in the first state, so that the outcome of the first preplay to be $C(\phi)$ with a non-negligible probability.

5 Conclusion

In this paper, we examined whether the n-person prisoners' dilemma with $n \geq 3$ is resolved as the outcome of a PEP via the preplay negotiation procedure proposed by Kalai (1981). We showed that if the preplay is executed at least once, then there is a PEP in which all players choose to cooperate in the final outcome. We also showed that if the preplay is executed at least twice, then there is a PEP in which all players choose to cooperate in the final outcome regardless of which entering action profile is given first. We conclude that Kalai's (1981) procedure resolves the n-person prisoners' dilemma for any $n \geq 2$.

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Appendix: Lemmas and Proofs

We define additional notations before providing the following lemmas and proofs. When the players' preplay strategies of the lth and subsequent preplays are given, for any $W \subseteq N$, we define $E^l_{iz}(W)$ as player i's expected payoff in Γ^k_z when $C(W)$ is the entering action profile of the lth preplay. For convenience of the notation, we define $E^{k+1}_{iz}(W)$ as player i's payoff when $C(W)$ is the final outcome. Further, for the terms in the expression of $E^l_{iz}(W)$, we use the simplified notation $E(W)$ as $E^{l+1}_{iz}(W)$.

We show four lemmas. The first lemma provides the expression of $E^l_{iz}(W)$, and the other three evaluate the differences in $E^l_{iz}(W)$ for various entering action profiles.

Lemma A1. *If players use preplay strategies $(\tau_1, ..., \tau_n)$ in Γ^k_z, then for any $l = 2, 3, 4, ..., k$, any $i \in N$ and any $W \subset N$, we have*

$$E^l_{iz}(W) = E(\phi) + p^1_W \sum_{j \in W} \{E(\{j\}) - E(\phi)\} + p^2_W \{E(W) - E(\phi)\} + o(\epsilon_z^{4k-1}),$$

where $p^1_W = \epsilon_z^3(1-\epsilon_z^2)^{|W|-1}(1-\epsilon_z^{4k})^{2(n-|W|)}$ if $|W| \geq 2$, $p^1_W = |W|\epsilon_z^3(1-\epsilon_z^{4k})^{n-1}$ if $|W| \leq 1$, $p^2_W = \epsilon_z^{2|W|}(1-\epsilon_z^{4k})^{n-|W|}$ if $|W| \geq 2$, and $p^2_W = 0$ if $|W| \leq 1$.

Proof. When $W = \phi$, the equality is obvious. When $W = \{j\}$, $C(\{j\})$ is the outcome of the lth preplay if C is realized during player j's first move, while D is realized during the other players' moves. Its probability is p_W^1. The outcomes other than $C(\phi)$ or $C(\{j\})$ occur with a probability of at most $O^+(\epsilon_z^{4k})$, hence $o(\epsilon_z^{4k-1})$. Therefore, we obtain the desired equality. Suppose that $|W| \geq 2$. When $C(W)$ is the entering action profile, the outcome of the lth preplay is $C(\phi)$ w.p.1 $z \to \infty$. because $W \subset N$. For any $j \in W$, the outcome is $C(\{j\})$ if C is realized twice during player j's moves, while D is realized during the other players' moves. Its probability is p_W^1. There are other cases in which C is realized three times or more during player j's move, while the other players in W gradually choose D, and $C(\{j\})$ finally becomes the outcome of this preplay. They occur with probability $o(\epsilon_z^{4k-1})$. The outcome is $C(W)$ if C is realized during the moves of all players in W, while D is realized during the other players' moves. Its probability is p_W^2. For any other $U \subset W$, the outcome is $C(U)$ with probability $o(\epsilon_z^{4k-1})$. Hence, we have the equality stated in the lemma. □

Lemma A2. *If players use preplay strategies $(\tau_1, ..., \tau_n)$ in Γ_z^k, then for any $l = 2, 3, 4, ..., k$, any $i \in N$, and any two subsets W and W' of $N - \{i\}$ such that $|W| > |W'|$, it holds that $E_{iz}^l(W) - E_{iz}^l(W') = O^+(\epsilon_z^{3(k-l+1)})$.*

Proof. Let $w = |W|$ and $w' = |W'|$. Lemma A1 provides the expression of $E_{iz}^l(W)$. Note that for any j and $j' \in N - \{i\}$, we have $E(\{j\}) = E(\{j'\})$. Hence, we have

$$E_{iz}^l(W) = E(\phi) + wp_W^1\{E(\{j\}) - E(\phi)\} + p_W^2\{E(W) - E(\phi)\} + o(\epsilon_z^{4k-1}),$$

where j is an arbitrary player in $N - \{i\}$. We have the similar expression for $E_{iz}^l(W')$. Therefore, we have

$$E_{iz}^l(W) - E_{iz}^l(W') = \{wp_W^1 - w'p_{W'}^1\}\{E(\{j\}) - E(\phi)\} + p_W^2\{E(W) - E(\phi)\}$$
$$-p_{W'}^2\{E(W') - E(\phi)\} + o(\epsilon_z^{4k-1}).$$

First, consider the case that $w' = 0$. We have

$$E_{iz}^l(W) - E_{iz}^l(W') = wp_W^1\{E(\{j\}) - E(\phi)\} + p_W^2\{E(W) - E(\phi)\} + o(\epsilon_z^{4k-1}).$$

We have $p_W^1 = O^+(\epsilon_z^3)$, $p_W^2 \leq O^+(\epsilon_z^4)$ and $E_{iz}^{k+1}(M) - E_{iz}^{k+1}(\phi) = f_i(D, |M|) - f_i(D, 0)$ for any $M \subseteq N - \{i\}$. Hence, we can obtain the desired equality by mathematical induction: for $l = k$, the equality holds; if the equality holds for $l = l'$, then it holds for $l = l' - 1$. Next, consider the case where $w' = 1$. In this case, $w \geq 2$. We have

$$E^l_{iz}(W) - E^l_{iz}(W') = (wp^1_W - p^1_{W'})\{E(\{j\}) - E(\phi)\} + p^2_W\{E(W) - E(\phi)\} + o(\epsilon_z^{4k-1}).$$

Here, $p^1_W = \epsilon_z^3 + o(\epsilon_z^4)$ and $p^1_{W'} = \epsilon_z^3 + o(\epsilon_z^{4k-1})$.[2] Hence, $wp^1_W - p^2_{W'} = O^+(\epsilon_z^3)$. Since $p^2_W = O^+(\epsilon_z^{2w})$, we have

$$E^l_{iz}(W) - E^l_{iz}(W') = O^+(\epsilon_z^3)\{E(\{j\}) - E(\phi)\} + O^+(\epsilon_z^{2w})\{E(W) - E(\phi)\} + o(\epsilon_z^{4k-1}).$$

We obtain the desired equality from the above result for $W' = \phi$. Finally, consider the case that $w' \geq 2$. We have $wp^1_W - w'p^1_{W'} = O^+(\epsilon_z^3)$. Further, p^2_W and $p^2_{W'}$ are $O^+(\epsilon_z^{2w})$ and $O^+(\epsilon_z^{2w'})$, respectively. Hence, we have

$$\begin{aligned} E^l_{iz}(W) - E^l_{iz}(W') = {} & O^+(\epsilon_z^3)\{E(\{j\}) - E(\phi)\} + O^+(\epsilon_z^{2w})\{E(W) - E(\phi)\} \\ & -O^+(\epsilon_z^{2w'})\{E(W') - E(\phi)\} + o(\epsilon_z^{4k-1}). \end{aligned}$$

We obtain the desired equality from the above result for $W' = \phi$. □

Lemma A3. *If players use preplay strategies $(\tau_1, ..., \tau_n)$ in Γ^k_z, then for any $l = 2, 3, 4, ..., k$, any $i \in N$, and any $W \subset N - \{i\}$, it holds that $E^l_{iz}(W) - E^l_{iz}(W \cup \{i\}) = O^+(\epsilon_z^{3(k-l+1)})$.*

Proof. Let $|W| = w$ and let $W_i = W \cup \{i\}$. First, consider the case $W = \phi$. By Lemma A1, we have $E^l_{iz}(\phi) = E(\phi) + o(\epsilon_z^{4k-1})$ and $E^l_{iz}(\{i\}) = E(\phi) + O^+(\epsilon_z^3)\{E(\{i\}) - E(\phi)\} + o(\epsilon_z^{4k-1})$. Hence, we have $E^l_{iz}(\phi) - E^l_{iz}(\{i\}) = O^+(\epsilon_z^3)\{E(\phi) - E(\{i\})\} + o(\epsilon_z^{4k-1})$. We can obtain the desired equality by mathematical induction. Next, consider the case $W = \{j\}$. From Lemma A1 we have

$$\begin{aligned} E^l_{iz}(W) &= E(\phi) + p^1_W\{E(\{j\}) - E(\phi)\} + o(\epsilon_z^{4k-1}), \\ E^l_{iz}(W_i) &= E(\phi) + p^1_{W_i}\{E(\{j\}) - E(\phi)\} + p^1_{W_i}\{E(\{i\}) - E(\phi)\} \\ & \quad +p^2_{W_i}\{E(\{i,j\}) - E(\phi)\} + o(\epsilon_z^{4k-1}). \end{aligned}$$

[2] We have this expression of $p^1_{W'}$ because the perturbation probability is ϵ_z^3 on the branch C. If it is ϵ_z^2, then $E^l_{iz}(W) - E^l_{iz}(W') < 0$ for sufficiently large z.

Since $E(\{i,j\}) - E(\phi) = E(\{i,j\}) - E(\{j\}) + E(\{j\}) - E(\phi)$, we have

$$E^l_{iz}(W) - E^l_{iz}(W_i) = (p^1_W - p^1_{W_i} - p^2_{W_i})\{E(\{j\}) - E(\phi)\} + p^1_{W_i}\{E(\phi) - E(\{i\})\}$$
$$-p^2_{W_i}\{E(\{i,j\}) - E(\{j\}\} + o(\epsilon_z^{4k-1}).$$

We can easily verify that $p^1_W - p^1_{W_i} = O^+(\epsilon_z^5)$. Hence, $p^1_W - p^1_{W_i} - p^2_{W_i} = -O^+(\epsilon_z^4)$ because $p^2_{W_i} = O^+(\epsilon_z^4)$. Since $p^1_{W_i} = O^+(\epsilon_z^3)$, we have

$$E^l_{iz}(W) - E^l_{iz}(W_i) = -O^+(\epsilon_z^4)\{E(\{j\}) - E(\phi)\} + O^+(\epsilon_z^3)\{E(\phi) - E(\{i\})\}$$
$$-O^+(\epsilon_z^4)\{E(\{j\}) - E(\{i,j\})\} + o(\epsilon_z^{4k-1}).$$

Hence, from the above result for $W = \phi$ and Lemma A2, we have

$$E^l_{iz}(W) - E^l_{iz}(W_i) = O^+(\epsilon_z^{3(k-l+1)}) - O^+(\epsilon_z^4)\{E(\{j\}) - E(\{i,j\})\} + o(\epsilon_z^{4k-1}).$$

We can obtain the desired equality by mathematical induction. Finally, consider the case $w \geq 2$. From Lemma A1 we have

$$E^l_{iz}(W) = E(\phi) + p^1_W \sum_{j \in W}\{E(\{j\}) - E(\phi)\} + p^2_W\{E(W) - E(\phi)\} + o(\epsilon_z^{4k-1}),$$

and the same one with W_i instead of W. Here, $p^1_W = g(\epsilon_z)(1-\epsilon_z^{4k})^2$ and $p^1_{W_i} = g(\epsilon_z)(1-\epsilon_z^2)$, where $g(\epsilon_z) = \epsilon_z^3(1-\epsilon_z^2)^{w-1}(1-\epsilon_z^{4k})^{2(n-w-1)}$. Hence, $p^1_W - p^1_{W_i} = O^+(\epsilon_z^5)$. Further, $p^2_W = O^+(\epsilon_z^{2w})$ and $p^2_{W_i} = O^+(\epsilon_z^{2(w+1)})$. Hence, we have

$$E^l_{iz}(W) - E^l_{iz}(W_i)$$
$$= O^+(\epsilon_z^5)\sum_{j \in W}\{E(\{j\}) - E(\phi)\} + O^+(\epsilon_z^3)\{E(\phi) - E(\{i\})\}$$
$$+O^+(\epsilon_z^{2w})\{E(W) - E(\phi)\} - O^+(\epsilon_z^{2(w+1)})\{E(W_i) - E(\phi)\} + o(\epsilon_z^{4k-1}).$$

We have $E(W_i) - E(\phi) = \{E(W_i) - E(W)\} + \{E(W) - E(\phi)\}$. By the above result for $W = \phi$ and Lemma A2, we have

$$E^l_{iz}(W) - E^l_{iz}(W_i) = O^+(\epsilon_z^{3(k-l)+3}) + O^+(\epsilon_z^{2(w+1)})\{E(W) - E(W_i)\} + o(\epsilon_z^{4k-1}).$$

We can obtain the desired equality by mathematical induction. □

Lemma A4. *If players use preplay strategies $(\tau_1, ..., \tau_n)$ in Γ^k_z, then for any $l = 2, 3, 4, ..., k$, any $i, j \in N$ $(i \neq j)$ and any $W \subset N - \{i,j\}$, it holds that $E^l_{iz}(W \cup \{i,j\}) - E^l_{iz}(W \cup \{i\}) = o(\epsilon_z^{3(k-l+1)-1})$.*

Proof. We have $E^l_{iz}(W \cup \{i,j\}) - E^l_{iz}(W \cup \{i\}) = \{E^l_{iz}(W \cup \{i,j\}) - E^l_{iz}(W \cup \{j\})\} + \{E^l_{iz}(W \cup \{j\}) - E^l_{iz}(W)\} + \{E^l_{iz}(W) - E^l_{iz}(W \cup \{i\})\}$. By Lemma A2 and A3, all the differences in the braces are $O(\epsilon_z^{3(k-l+1)})$. Since one is negative and two are positive, the term $\epsilon_z^{3(k-l-1)}$ may vanish. Hence, we obtain the desired equality. □

In the following, we prove Proposition 1, Theorem 2 and 3. In the proofs, it is often necessary to state "with a probability that converges to 1 when $z \to \infty$" about the occurrence of outcomes of preplays. We describe "w.p.1 $z \to \infty$" as an abbreviation for it.

Proof of Proposition 1. We show that $(\tau_1, ..., \tau_n)$ is a subgame perfect equilibrium of $\Gamma^k_z(e)$ if z is sufficiently large. To demonstrate this, it suffices to show that for any $l = 1, 2, 3, ..., k$, any $i \in N$ and any $P \in \mathscr{P}^l_i$, player i obtains a more expected payoff by choosing $\tau_i(P)$ than by choosing $b \neq \tau_i(P)$ in P if z is sufficiently large, when every player $j \in I(P) - \{i\}$ chooses $\tau_j(P')$ in the information set $P' \in \mathscr{P}^l_j$ such that $(a(P'), I(P')) = (a(P), I(P))$, and when all players make choices by $(\tau_1, ..., \tau_n)$ for all information sets after $(a(P), I(P))$. Let $(a(P), I(P)) = (a, I)$. To make the statements shorter, we say that a *is better than* b *for player* i, indicating that player i obtains a more expected payoff by choosing an action a than by choosing an action b. We consider the following four cases.

(1) The case that $I = \{i\}$.

If $l = k$, then, based on Assumption A1, $D = \tau_i(P)$ is better than C for player i for any z. Assume $l \leq k-1$. Consider the case $a_{N-\{i\}} = C(N - \{i\})$. If player i chooses $C = \tau_i(P)$, then the final outcome is $C(N)$ w.p.1 $z \to \infty$. If player i chooses D, then the final outcome is $C(\phi)$ w.p.1 $z \to \infty$. Hence, based on Assumption A2, C is better than D for player i if z is sufficiently large. Consider the case $a_{N-\{i\}} \neq C(N - \{i\})$. Let $W = (N - \{i\})_{aC}$. If player i chooses $D = \tau_i(P)$, then the outcome of this preplay is $C(W)$ w.p.1 $z \to \infty$. If player i chooses C, then the outcome of this preplay is $C(W \cup \{i\})$ w.p.1 $z \to \infty$. By Lemma A3, $E^{l+1}_{iz}(W) - E^{l+1}_{iz}(W \cup \{i\}) = O^+(\epsilon_z^{3(k-l)})$, hence C is better than D for player i if z is sufficiently large.

(2) The case where $(a, I) = (C(N), N)$.

If player $i \in N$ chooses $C = \tau_i(P)$, then the final outcome is $C(N)$ w.p.1 $z \to \infty$. If player i chooses D, then the final outcome is $C(\phi)$ w.p.1 $z \to \infty$. Hence, based on Assumption A2, C is better than D for player i if z is sufficiently large.

(3) The case that $a \neq C(N)$ and $I = N$.

Consider the case $l = 1$. If player $i \in I_{aD}$ chooses $D = \tau_i(P)$, then the outcome of the first preplay is $C(\phi)$ w.p.1 $z \to \infty$. Any other outcome occurs with probability $o(\epsilon_z^{4k-1})$. If player i chooses C, then the outcome is $C(\{i\})$ w.p.1 $z \to \infty$. Any other outcome occurs with probability $o(\epsilon_z^{4k-1})$. Hence, by Lemma A3, D is better than C for player i if z is sufficiently large. If player $i \in I_{aC}$ chooses $D = \tau_i(P)$, then the outcome of the first preplay is $C(\phi)$ w.p.1 $z \to \infty$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Assume $|I_{aC}| = 1$. If player i chooses C, then the outcome of the first preplay is $C(\{i\})$ w.p.1 $z \to \infty$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Hence, by Lemma A3, D is better than C for player i if z is sufficiently large. Assume $|I_{aC}| \geq 2$. If player i chooses C, then the outcome of the first preplay is $C(\phi)$ w.p.1 $z \to \infty$. However, the outcome is $C(\{i\})$ with a probability of $O^+(\epsilon_z)$. This occurs when C is realized twice during player i's move, while D is realized during the other players' moves. Any other outcome occurs with probability of $o(\epsilon_z^{4k-1})$. Hence, by Lemma A3, D is better than C for player i.

Consider the case $l \geq 2$. Let $I_{aC} = W$. If player $i \in N$ chooses $D = \tau_i(P)$, then it is the case that all players make choices by $(\tau_1, ..., \tau_n)$ during the first move of the lth preplay. Hence, player i's expected payoff E_{D_1} is equal to $E_{iz}^l(W)$. In its expression, which is given in Lemma A1, $E(\{j\}) - E(\phi)$ and $E(W) - E(\phi)$ are positive by Lemma A2. Hence, we have $E_{D_1} \geq E(\phi) + o(\epsilon_z^{4k-1})$. Suppose that player $i \in I_{aD}$ chooses C. Then the outcome of the lth preplay is $C(\{i\})$ w.p.1 $z \to \infty$. Hence, if $l = k$, then D is better than C based on Assumption A1. Assume $l \leq k - 1$. For each $j \in W$, let p_j be the probability that the outcome of the lth preplay is $C(\{i, j\})$. For any $j, j' \in W$, we have $p_j = p_{j'}$. For any $j \in W$, we have $p_j = O^+(\epsilon_z^3)$ if $|W| \geq 2$ and $p_j = O^+(\epsilon_z^4)$ if $|W| = 1$. Because $C(\{i, j\})$ occurs when C is realized twice during player j's moves while the other players choose

D, and any other state transition through which $C(\{i,j\})$ realizes occurs with probability $o(\epsilon_z^{4k-1})$. Any other outcome occurs with probability $o(\epsilon_z^{4k-1})$. Hence, player i's expected payoff is

$$\begin{aligned}
E_{C_1} &= E(\{i\}) + \sum_{j\in W} p_j\{E(\{i,j\}) - E(\{i\})\} + o(\epsilon_z^{4k-1}) \\
&= E(\{i\}) + o(\epsilon_z^{3(k-l)+2}).
\end{aligned}$$

We have the second equality by Lemma A4. Since $E(\phi) - E(\{i\}) = O^+(\epsilon_z^{3(k-l)})$ based on Lemma A3, we have $E_{D_1} > E_{C_1}$ if z is sufficiently large.

Suppose that player $i \in I_{aC} = W$ chooses C. If $W = \{i\}$, then the outcome of the lth preplay is $C(\{i\})$ w.p.1 $z \to \infty$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Hence, player i's expected payoff is $E(\{i\}) + o(\epsilon_z^{4k-1})$. Therefore, based on Lemma A3, D is better than C for player i if z is sufficiently large. If $|W| \geq 2$, then the outcome of the lth preplay is $C(\phi)$ w.p.1 $t \to \infty$. The outcome is $C(\{i\})$ with a probability of $O^+(\epsilon_z)$, which is the probability that C is realized during the move of player i in the next state. The outcome is $C(W)$ with a probability of $O^+(\epsilon_z^{2(|W|-1)})$, which is the probability that C is realized during all the moves of the players in $W - \{i\}$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Hence, player i's expected payoff is

$$E_{C_2} = E(\phi) + O^+(\epsilon_z)\{E(\{i\}) - E(\phi)\} + O^+(\epsilon_z^{2(|W|-1)})\{E(W) - E(\phi)\} + o(\epsilon_z^{4k-1}).$$

By Lemmas A2 and A3, we have $E_{C_2} = E(\phi) - O^+(\epsilon_z^{3(k-l)+1})$. Hence, we have $E_{D_1} > E_{C_2}$ if z is sufficiently large.

(4) The case that $a \neq C(N)$ and $2 \leq |I| \leq n-1$.

Let $W = (N - I)_{aC}$. We consider the following two sub-cases.

(4-1) The case that $|I_{aC}| = 1$.

Let $I_{aC} = \{j\}$. Suppose that player $i \in I_{aD}$ chooses $D = \tau_i(P)$. The outcome of the lth preplay is $C(W)$ w.p.1 $z \to \infty$. The outcome is $C(W \cup \{j\})$ with a probability of $O^+(\epsilon_z)$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Hence, player i's expected payoff is

$$E_{D_2} = E(W) + O^+(\epsilon_z)\{E(W \cup \{j\}) - E(W)\} + o(\epsilon_z^{4k-1}) \geq E(W).$$

We have the last inequality from Lemma A2. Suppose that player i chooses C. Then the outcome of the lth preplay is $C(W \cup \{i\})$ w.p.1 $z \to \infty$. If $l = k$, then based on Assumption A1, D is better than C for player i if z is sufficiently large. Assume $l \leq k-1$. The outcome is $C(W \cup \{i, j\})$ with a probability of $O^+(\epsilon_z)$. This is the probability that C is realized once during player i's move and twice during player j's move, while players in $I_{aD} - \{i\}$ choose D. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Hence, player i's expected payoff is

$$\begin{aligned} E_{C_3} &= E(W \cup \{i\}) + O^+(\epsilon_z)\{E(W \cup \{i, j\}) - E(W \cup \{i\})\} + o(\epsilon_z^{4k-1}) \\ &= E(W \cup \{i\}) + O(\epsilon_z^{3(k-l)+1}). \end{aligned}$$

We have the last equality from Lemma A4. Since $E(W) - E(W \cup \{i\}) = O^+(\epsilon_z^{3(k-l)})$ based on Lemma A3, we have $E_{D_2} > E_{C_3}$ if z is sufficiently large.

Let us examine the choice of player $i \in I_{aC}$. Now, $I_{aC} = \{i\}$. Suppose that player i chooses $D = \tau_i(a, I)$. The outcome of the lth preplay is then $C(W)$ w.p.1 $z \to \infty$. The outcome is $C(W \cup \{i\})$ with a probability of $O^+(\epsilon_z)$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Suppose that player i chooses C. The outcome of the lth preplay is $C(W \cup \{i\})$ w.p.1 $z \to \infty$, and any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Hence, according to Lemma A3, D is better than C for player i if z is sufficiently large.

(4-2) The case that $|I_{aC}| \geq 2$.

Suppose that player $i \in I_{aD}$ chooses $D = \tau_i(P)$. The outcome of the lth preplay is $C(W)$ w.p.1 $z \to \infty$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Suppose that player i chooses C. The outcome of the lth preplay is $C(W \cup \{i\})$ w.p.1 $z \to \infty$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. If $l = k$, based on Assumption A1, D is better than C for player i if z is sufficiently large. If $l \leq k-1$, based on Lemma A3, D is better than C for player i if z is sufficiently large.

Suppose that player $i \in I_{aC}$ chooses $D = \tau_i(P)$. The outcome of the lth preplay is then $C(W)$ w.p.1 $z \to \infty$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Suppose that player i chooses C. The outcome of the lth preplay is $C(W)$ w.p.1 $z \to \infty$. The outcome is $C(W \cup \{i\})$ with a probability of $O^+(\epsilon_z)$. This happens when C is realized during the move of player i twice, while D is realized during the other players' moves. It occurs through other state transitions, but their probabilities are $o(\epsilon_z^{4k-1})$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Based on Lemma A3, D is better than C for player i if z is sufficiently large. $\square$

Proof of Theorem 2. We show that $(\sigma_1, ..., \sigma_n)$ is a subgame perfect equilibrium of $\Gamma_z^{ke}(e')$ if z is sufficiently large. To demonstrate this, it suffices to show that for any $l = 1, 2, 3, ..., k$, any $i \in N$ and any $P \in \mathscr{P}_i^l$, player i obtains a more expected payoff by choosing $\sigma_i(P)$ than by choosing $b \neq \sigma_i(P)$ in P if z is sufficiently large, when every player $j \in I(P) - \{i\}$ chooses $\sigma_j(P')$ in the information set $P' \in \mathscr{P}_j^l$ such that $(a(P'), I(P')) = (a(P), I(P))$, and when all players make choices by $(\sigma_1, ..., \sigma_n)$ in all information sets after $(a(P), I(P))$. From Proposition 1, it is sufficient to show this proposition for $l = 1$. Let $(a(P), I(P)) = (a, I)$. Similar to the above proof, we say that *a is better than b for player i*, indicating that player i obtains a more expected payoff by choosing an action a than by choosing an action b. If $I(P) = \{i\}$, then the proof is the same as the proof of Proposition 1 in the case (1). Thus, we consider the case $|I(P)| \geq 2$.

(1) The case that $(N - I)_{aD} = \phi$.

If player $i \in I_{aC}$ chooses $C = \sigma_i(P)$, then the final outcome is $C(N)$ w.p.1 $z \to \infty$. If player i chooses D, then the final outcome is $C(\phi)$ w.p.1 $z \to \infty$. Hence, based on Assumption A2, C is better than D for player i if z is sufficiently large.

If player $i \in I_{aD}$ chooses $C = \sigma_i(P)$, then the final outcome is $C(N)$ w.p.1 $z \to \infty$. Any other outcome occurs in the first preplay with a probability of $o(\epsilon_z^{4k-1})$. If player i chooses D, then the outcomes are as follows. When $I_{aD} = \{i\}$, the outcome of the first preplay is $C(N - \{i\})$, so the final outcome is $C(\phi)$ w.p.1 $z \to \infty$. Based on Assumption A2, C is better than D for player i if z is sufficiently large. When $I_{aD} \geq 2$, the outcome of the first preplay is $C(N)$ w.p.1 $z \to \infty$. The outcome is $C(N - \{i\})$ with a probability of $O^+(\epsilon_z)$. Hence, with this probability, the final outcome is $C(\phi)$. Any other outcome occurs in the first preplay with a probability of $o(\epsilon_z^{4k-1})$. Therefore, based on Assumption A2, C

is better than D for player i if z is sufficiently large.

(2) The case where $(N-I)_{aD} \neq \phi$.

Let $W = (N-I)_{aC}$. If player $i \in I_{aC}$ chooses $D = \sigma_i(P)$, then the outcome of the first preplay is $C(W)$ w.p.1 $z \to \infty$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. If player i chooses C, then the outcome is as follows. When $I_{aC} = \{i\}$, then the outcome is $C(W \cup \{i\})$ w.p.1 $z \to \infty$. When $|I_{aC}| \geq 2$, the outcome of the first preplay is $C(W)$ w.p.1 $z \to \infty$, but the outcome is $C(W \cup \{i\})$ with a probability of $O^+(\epsilon_z)$. This occurs when C is realized during player i's move twice, while D is realized during the moves of the other players. It also occurs in other state transitions, but their probabilities are $o(\epsilon_z^{4k-1})$. Any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Hence, based on Lemma A3, D is better than C for player i if z is sufficiently large.

Suppose that player $i \in I_{aD}$ chooses $D = \sigma_i(P)$. The outcome of the first preplay is then $C(W)$ w.p.1 $z \to \infty$. If $|I_{aC}| \neq 1$, any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. If $I_{aC} = \{j\}$, then the outcome is $C(W \cup \{j\})$ with a probability of $O^+(\epsilon_z)$, and any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Hence, player i's expected payoff is $E_D = E_{iz}^2(W) + \alpha + o(\epsilon_z^{4k-1})$, where $\alpha = O^+(\epsilon_z)\{E_{iz}^2(W \cup I_{aC}) - E_{iz}^2(W)\}$ if $|I_{aC}| = 1$, and otherwise $\alpha = 0$. Based on Lemma A2, we have $\alpha = O^+(\epsilon_z^{3(k-1)+1})$ if $|I_{aC}| = 1$. Suppose that player i chooses C. Then, the outcome of the first preplay is $C(W \cup \{i\})$ w.p.1 $z \to \infty$. If $|I_{aC}| \neq 1$, any other outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. If $I_{aC} = \{j\}$, then the outcome of the first preplay is $C(W \cup \{i, j\})$ with a probability of $O^+(\epsilon_z^2)$. This is the probability that C is realized twice during the moves of player j. Any other outcome occurs with probability $o(\epsilon_z^{4k-1})$. Hence, player i's expected payoff is $E_C = E_{iz}^2(W \cup \{i\}) + \beta + o(\epsilon_z^{4k-1})$, where $\beta = O^+(\epsilon_z^2)\{E_{iz}^2(W \cup I_{aC} \cup \{i\}) - E_{iz}^2(W \cup \{i\})\}$ if $|I_{aC}| = 1$, and $\beta = 0$ otherwise. By Lemma A4, we have $\beta = O(\epsilon_z^{3(k-1)+2})$ when $|I_{aC}| = 1$. Hence, we have $\alpha = \beta = 0$ or $\alpha > \beta$ if z is sufficiently large. By Lemma A3, we have $E_{iz}^2(W) - E_{iz}^2(W \cup \{i\}) = O^+(\epsilon_z^{3(k-1)})$. Therefore, we have $E_D > E_C$ if z is sufficiently large. $\square$

Proof of Theorem 3. From Theorem 2, it is sufficient to show that for any $i \in N$, player i obtains a more expected payoff in Γ_z^{ke} by choosing e_i than by choosing $b \neq e_i$ at P_i^0 if z is sufficiently large, when every player $j \in N - \{i\}$ chooses e_j at P_j^0, and all players make choices by $(\sigma_1, ..., \sigma_n)$ in all information sets in the preplays. If player i chooses e_i, then the outcome of the first preplay is $C(N)$, so the outcome of the final preplay is $C(N)$ w.p.1 $z \to \infty$. Any other outcome occurs in the first preplay with a probability of $o(\epsilon_z^{4k-1})$. If player i chooses $b \neq e_i$, the outcome of the first preplay is $C(N)$ w.p.1 $z \to \infty$. Hence, the outcome of the final preplay is similar. However, the outcome of the first preplay is $C(N - \{i\})$ with a probability of $O^+(\epsilon_z)$ if $e = C(N)$ or $C(\phi)$, and is otherwise $O^+(\epsilon_z^2)$. Here, $O^+(\epsilon_z)$ is the probability that D is realized during the move of player i while C is realized during the other players' moves, and $O^+(\epsilon_z^2)$ is the probability that D is realized twice during the move of player i while C is realized during the other players' moves once or twice. With these probabilities, the final outcome is $C(\phi)$. Any other final outcome occurs with a probability of $o(\epsilon_z^{4k-1})$. Therefore, based on Assumption A2, player i obtains a more expected payoff by choosing e_i than by choosing $b \neq e_i$ if z is sufficiently large. $\square$

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