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Abstract

It has been proven by Hayashi (1982) that Tobin's average *q* is equal to Tobin's marginal *q* under certain conditions, without considering replacement investment. Here we show that Tobin's average *q* is larger than Tobin's marginal *q* when the option of replacement investment is evaluated in a share market; additionally, the first theorem in Miller and Modigliani (1961) does not hold because equity financing increases the value of a firm. We also obtained a closed form solution for Tobin' average *q* and marginal *q*, and found that marginal *q* can be inferred from average *q* given that the former cannot be observed directly. These results are robust in a capital market with a lease contract. Our theory could explain why average *q* is larger than marginal *q* in the existing empirical literature, and why some firms prefer equity financing.

JEL codes: D21, D24

Keywords: Average *q*, Marginal *q*, MM theory, lease, replacement investment

1 Introduction

1.1 Background and research questions

Limited investment opportunity is the key constraint for corporate profit maximization. For example, when a new building is planned, the first problem is the decision regarding when the existing building in that location should be demolished. This is similar to the situation of a firm scrapping old equipment so that it can be replaced with new equipment. In traditional investment theory, a concave production function with physical capital investment is often assumed, which is equivalent to the decreasing marginal efficiency of capital in Keynes (1936). This assumption implies that there are enough investment opportunities with decreasing marginal productivity, while replacement investment "by scrap" is not explicitly considered. As pointed out by Feldstein and Rothschild (1974), over half of the investment in the United States is replacement investment. Therefore, explicit analysis of replacement is important for investment decision theory, as argued by Hotelling (1925), Preinreich (1940), and Bellman (1955).

Regarding real investment, Tobin's q theory has been used as the core analytic tool for both micro firms and the macroeconomy for the past half century.[3]

[3] See Bartlett and Partnoy (2020) for a discussion of Tobin's q theory and its applications, and see Jorgenson (1963, 1965) for the alternative neoclassical real investment theory.

Considerable progress has been made since Hayashi (1982) showed a direct link between Tobin's average *q* and Tobin's marginal *q*, where the latter should be more important when considering new marginal investments, even though it cannot be observed directly. Hence, empirically estimating marginal *q* has become an important and challenging task for applied analysis of real investment (Abel and Blanchard 1986; Ogawa and Kitasaka 1999; Gala, Gomes and Liu 2022; Wan and Qiu 2022). Previous studies have shown that the estimated marginal *q* is usually lower than the estimated average *q*. One possible reason for this is that average *q* could include a price bubble in the stock market (real capital) (Ogawa and Kitasaka 1999; Chirinko and Schaller 2001).

In this study, we explore how Tobin's average *q* and Tobin's marginal *q* change when explicitly considering replacement investment. We will show that average *q* could be higher than marginal *q* even without an asset bubble, by explicitly incorporating the replacement decision presented by Hotelling (1925) into the *q* theory of Tobin (1969) and Hayashi (1982). From this, we can infer the directly unobservable marginal *q* from the directly observable average *q*. Furthermore, we discuss whether the first theorem in Miller and Modigliani (1961), hereinafter referred to as the first MM theorem, holds under the opportunity of replacement investment in the future and then discuss why some firms prefer equity financing to debt financing, as argued by Stiglitz (1988).

1.2 Contribution of the paper

As proven by Hayashi (1982), Tobin's average q is equal to Tobin's marginal q under certain conditions when replacement investment is not considered. Here, we show that Tobin's average q is higher than Tobin's marginal q when considering replacement investment, and that the first MM theory does not hold because equity financing increases the value of a firm. Using closed form solutions for Tobin's average q and marginal q, the latter must be inferred because it cannot be observed directly.

1.3 Organization of the paper

Section 2 introduces a replacement investment model to analyze the relationship between Tobin's average q and Tobin's marginal q, and examines whether the first MM theorem holds. Section 3 further examines Tobin's average q and marginal q, as well as the first MM theorem, when considering replacement investment in a lease market. Finally, we summarize our findings in Section 4 and discuss some of the issues left for future research.

2 Replacement investment

2.1 The basic model

We consider that an investor makes decisions regarding the timing of a new investment, such as the purchase of a new building, machinery, or equipment, and its replacement after utilizing the purchased item for some time (τ), following Wan (2019). The price or cost of the investment is assumed to be I_0 initially (at time zero) and the interest rate (r) is set as a constant. The investment is considered to be a sufficiently long term; its marginal productivity or dividend (d) decreases with time at a constant rate (δ). This is because the initial investment I_0 (> 0) depreciates over time at a constant rate (δ). Thus, the production function is linear [$d = \alpha I_0, \alpha > 0, d_t = \alpha(I_0 e^{-\delta t})$]. An investor will obtain the discounted value of net profit ($\pi(\tau)$) at time zero, as follows:

$$\begin{aligned}\pi(\tau) &= \int_0^{\tau} \alpha[I_0 e^{-\delta t}][e^{-rt}]dt - I_0 \\ &= \int_0^{\tau}\left[(\alpha I_0)e^{-\delta t}\right][e^{-rt}]dt - I_0 \\ &= \int_0^{\tau}\left[de^{-\delta t}\right][e^{-rt}]dt - I_0 \\ &= \frac{d}{r+\delta}\left[1 - e^{-(r+\delta)\tau}\right] - I_0. \end{aligned} \quad (1)$$

The necessary condition for positive profit is a strictly positive value for Eq. (1), which is unchanging with respect to the time of usage of equipment (τ). Without replacement of the equipment, its value converges to

$$\lim_{\tau \to \infty} \pi(\tau) = \frac{d}{r+\delta} - I_0. \quad (2)$$

Note $\frac{d}{(r+\delta)I_0}\left(=\frac{\alpha I_0}{(r+\delta)I_0}=\frac{\alpha}{r+\delta}=q\right)$ in Eq. (2) is equal to Tobin's average q and marginal q (Tobin 1969); a ratio with a value exceeding 1 ($q > 1$ if and only if $\alpha > r + \delta$) ensures a net positive profit. Here it is assumed that q is larger than 1, and that average q $\left(=\frac{\alpha I_0/(r+\delta)}{I_0}\right)$ is equal to marginal q $\left(=\frac{\alpha}{r+\delta}\right)$, as shown in Hayashi (1982), because the dividend is a linear homogeneous function of the net capital stock.

Consider financing the fund for I_0 via share/s (S) or debt (D), as shown in Figure 1.

Property 1:

The first MM theorem holds, i.e., the structure of the capital is not relevant to the value of the firm.

Proof:

The value of a firm is defined by its net wealth on the balance sheet. We analyze the following three cases.

Case 1: Debt financing

$$A = \frac{\alpha}{r+\delta} I_0, \tag{3}$$

$$D = I_0, \tag{4}$$

$$Net\ wealth = \frac{\alpha}{r+\delta} I_0 - I_0. \tag{5}$$

Case 2: Share financing

$$A = \frac{\alpha}{r+\delta} I_0,$$

$$Book\ value\ of\ share = I_0, \tag{6}$$

$$Net\ wealth = Market\ value\ of\ share - book\ value\ of\ share = \frac{\alpha}{r+\delta} I_0 - I_0. \tag{7}$$

Case 3: Mixed financing by a ratio β with $0 < \beta < 1$

$$A = \frac{\alpha}{r+\delta} I_0,$$

$$D = \beta I_0, \tag{8}$$

$$Book\ value\ of\ share = (1-\beta) I_0, \tag{9}$$

$$Market\ value\ of\ share = A - \beta I_0, \tag{10}$$

$$Net\ wealth = Market\ value\ of\ share - book\ value\ of\ debt \tag{11}$$

$$= (A - \beta I_0) - [(1 - \beta I_0)$$

$$= \frac{\alpha}{r+\delta} I_0 - I_0.$$

For the above three cases, the net wealth of the firm is the same as $\frac{\alpha}{r+\delta} I_0 - I_0$.

Q.E.D.

2.2 Valuation on replacement investment

We assume that the investor can maximize the net profit per period (τ). Also following Wan (2019), the investor considers the following problem:

$$\max_{0<\tau<\infty} \pi(\tau) = \sum_{n=0}^{\infty} \int_{n\tau}^{(n+1)\tau} \{[de^{-\delta t}][e^{-rt}]dt - I_0\}$$

$$= \frac{1}{1-e^{-r\tau}} \left\{ \frac{d}{(r+\delta)} \left[1 - e^{-(r+\delta)\tau}\right] - I_0 \right\}, \quad (12)$$

where, n = 0, 1, 2, ….

Under the assumption of irreversibility,[4] the salvage value at timing τ^* becomes zero.

From this, we obtain the following theorems.[5]

Theorem 1:

Under the replacement investment at timing τ^*, Tobin's average q is higher than Tobin's marginal q.

Proof*:*

Set Tobin's marginal $q = \frac{\alpha}{r+\delta} \equiv \underline{q}$ and Tobin's average $q \equiv \bar{q}$.

We obtain

$$\underline{q}(\tau^*) = 2\underline{q}^2 - 1 < \underline{q} \equiv \frac{\alpha}{r+\delta} \text{ for } \tau^*, \quad (13)$$

$$\bar{q} = \underline{q} + \left(\underline{q}^{\frac{1}{2}} - 1\right)^2 > \underline{q} \equiv \frac{\alpha}{r+\delta} > 1 \text{ for } n\tau^* \text{ and n}\to\infty, \quad (14)$$

$$\text{and } \underline{q} = \frac{1}{2}\bar{q} + \frac{1}{2}(2\bar{q} - 1)^{\frac{1}{2}}. \quad (15)$$

Q.E.D.

[4] There are many studies relevant to this assumption; see Abel and Eberly (1994), Arrow (1968), and Bernanke (1983) for details.

[5] The optimal timing is also referred to as the economic depreciation hypothesis; see Wan and Qiu (2022) for details.

Theorem 2:

Under the replacement investment, the value of the firm with equity financing is higher than its value with debt financing; thus, the first MM theorem does not hold.

Proof*:*

The value of the firm with debt financing = $\frac{\alpha}{r+\delta} I_0 = \underline{q} I_0$, while the firm's value with equity financing is $\bar{q} I_0$. By ***Theorem 1***, we obtain $\bar{q} I_0 > \underline{q} I_0$.

Q.E.D.

2.3 Technological progress

Next, we consider the impact of technological change, as argued by Boucekkine, del Rio, and Martinez (2009), on the optimal timing of replacement (τ^*). To express the economic meaning of technological innovation, we consider that innovation occurs if there is a ceteris paribus increase in the dividend (*d*). Then we obtain the following.

Proposition 1:

The difference between Tobin's average *q* and marginal *q*, as well as the

difference in the value of a firm between equity financing and debt financing, becomes larger when technological progress is expected.

***Proof**:*

By definition $\frac{\partial \underline{q}}{\partial \alpha} > 0$, then we obtain

$$\frac{\partial \bar{q}}{\partial \alpha} = \frac{\partial \bar{q}}{\partial \underline{q}} \frac{\partial \underline{q}}{\partial \alpha} = \left[1 + \left(\underline{q}^{\frac{1}{2}} - 1\right) \underline{q}^{-\frac{1}{2}}\right] \frac{\partial \underline{q}}{\partial \alpha} > 0, \tag{16}$$

$$\frac{\partial (\bar{q} - \underline{q})}{\partial \alpha} = \left(\underline{q}^{\frac{1}{2}} - 1\right) \underline{q}^{-\frac{1}{2}} > 0. \tag{17}$$

Q.E.D.

2.4 Impacts of tax and subsidy

We consider a tax or subsidy with a rate ($-1 \leqq \theta \leqq \frac{\alpha - (r+\delta)}{\alpha}$) on the dividend. Then we obtain the following.

Proposition 2:

The difference between Tobin's average *q* and marginal *q*, as well as the difference in the value of a firm between equity financing and debt financing, becomes smaller (larger) when taxes (subsidies) are introduced on dividends.

***Proof**:*

After a tax ($\theta > 0$) or subsidy ($\theta < 0$),

$$\underline{q} = \frac{(1-\theta)\alpha}{r+\delta}, \frac{\partial \underline{q}}{\partial \theta} < 0, \tag{18}$$

Then, ***Proposition 2*** is obtained by ***Proposition 1***.

Q.E.D.

3 Replacement in a lease market

3.1 *q* theory with replacement in a lease market

We now examine what happens when the institution of land is constrained by a limited period L ($0 < L \leqq \bar{L} \ll \infty$) via a lease, as shown in Wan (2018). It is assumed that a fixed cost ($\underline{c} = \eta I_0 > 0$) is required for a contract of re-lease. By calculating the optimal timing for replacement, we obtain the following:

Proposition 3:

The difference between Tobin's average q and marginal q becomes smaller when a fixed cost is necessary for a re-lease contract.

Proof*:*

$$\lim_{\tau \to \infty} \pi(\tau) = \frac{d}{r+\delta} - I_0 - \eta I_0 = \frac{\alpha I_0}{r+\delta} - (1+\eta) I_0, \tag{19}$$

Then, we obtain a marginal q with a re-lease cost:

$$\underline{q'} = \frac{\alpha}{(r+\delta)(1+\eta)} \; and \; \frac{\partial \underline{q'}}{\partial \eta} < 0. \tag{20}$$

Thus, ***Proposition 3*** is obtained by ***Proposition 1***.

Q.E.D.

3.2 MM theory with replacement in a lease market

The re-lease cost is assumed to be a positive fixed cost $\underline{c}$ $(= \eta I_0 > 0)$. This cost is considered in the cost of purchasing capital goods. The fixed cost lowers both marginal q and average q. Whether the first MM theorem holds depends on the size of the fixed cost. The net profit needs to cover the fixed cost $\underline{c}$; thus, the necessary condition for investment is

$$\frac{\alpha}{r+\delta} - (1+\eta) > 0.$$

From this, we obtain the following.

Proposition 4:

The first MM theorem holds when the re-lease cost is large; conversely, it does not hold when the re-lease cost is small.

Proof*:*

$$\frac{\alpha}{r+\delta} - \eta \to 1 \; for \; \eta \to \frac{\alpha}{r+\delta} - 1. \tag{21}$$

Under Equation (21), the first MM theorem holds. Conversely,

$$\frac{\alpha}{r+\delta} - \eta \rightarrow \frac{\alpha}{r+\delta} \; for \; \eta \rightarrow 0. \tag{22}$$

Under Equation (22), the first MM theorem does not hold by ***Theorem 2***.

Q.E.D.

4 Conclusion and future research

Hayashi (1982) proved that Tobin's average q is equal to Tobin's marginal q under certain conditions when replacement investment is not considered. By contrast, Tobin's average q is higher than Tobin's marginal q when future replacement investment is evaluated; additionally, the first MM theory does not hold because equity financing increases the value of a firm compared with debt financing. If closed form solutions for Tobin's average q and marginal q are obtained, the latter can be inferred from the former given that marginal q cannot be observed directly. The main results here are robust in a capital market with lease contracts. Our theory provides a possible explanation as to why average q is significantly larger than marginal q, and why some firms prefer equity financing to debt financing in the empirical literature.

The following research is left for future work. The replacement investment equation with the q ratio needs to be derived, along with the aggregation of replacement investment with different timing needs. Furthermore, different capital goods should

have different replacement timing; thus, multiple capital goods with multiple q should be analyzed with explicit consideration of the replacement investment. Finally, empirical studies with micro and macro data are necessary to test the theoretical predictions presented in this study.

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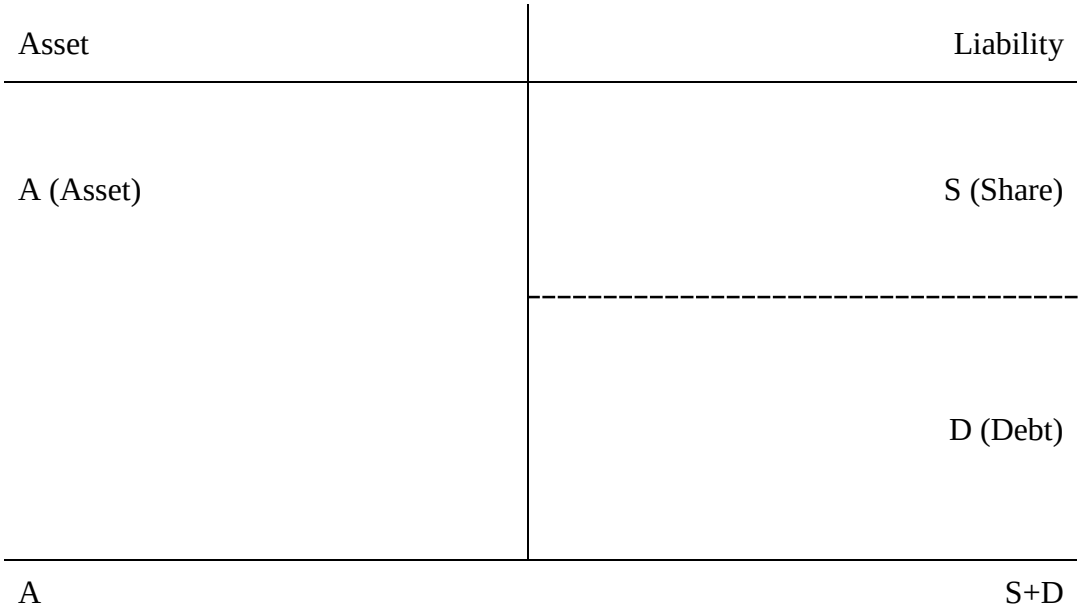
Figure 1: Balance sheet of a firm.



Source: Drawn by the author.